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Casimir energies of spherical plasma shells

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Abstract

We study a model loosely inspired by the giant carbon molecule C_{60}, namely a negligibly thin spherical shell of radius R, carrying a continuous fluid with mass and charge surface-densities m/a^2 and e/a^2 (plus an inert overall-neutralizing charge distribution), so that a mimics some mean inter-electron spacing, comparable say to the Bohr radius. The Casimir energy B is the total zero-point energy of the exact multipolar normal modes, minus that of empty space, minus the self-energy of the given amount of material at infinite dilution. Subject to a Debye-type cutoff $l \leqslant L$ on angular momenta l, but needing no frequency cutoff, B is a well-defined function of R, $x \equiv e^2/mc^2a$, and $X \equiv R/a$, expressible in terms of the multipolar phase shifts $\delta_l^{\mathrm{TE,TM}}$. We consider it only for $X \gg 1 \Rightarrow L \sim X$. Realistically one has $x \ll 1$, but $\mu \equiv 4\pi x X$ can be large or small. Then B is always dominated by terms of order $\hbar\sqrt{e^2/ma^3}X^2$ stemming from TM modes; but the pattern of corrections as functions of x and X is intricate, and accessible only through the Debye (uniform) expansions of the Bessel functions figuring in the δ_l. Historical interest attaches to *Boyer components*, far-subdominant parts of B having the form $(\hbar c/R)C_B$, where C_B is a pure number. When $\mu \ll 1$ (as in C_{60}) there are none, because all corrections are at least of order $x^{1/2}$, and none are proportional to $1/R$. But a Boyer component does exist when $\mu \gg 1$ (as in macroscopic shells), with $C_B = [3/64] - [(9/4096)(\pi^2/8 - 1)] + \cdots \simeq 0.0464$. The two terms come from orders 1 and 2 of the Debye expansion; the contribution from order 0 vanishes because of an apparently fortuituous cancellation between TE and TM. The most precise value proposed so far is $C_B = 0.046\,1765$; but the significance of comparisons is unclear, because previous calculations mistakenly treat the Boyer component as if it included all of B in a hypothetical perfect-reflector limit $x \to \infty$.

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1. Introduction

Suppose it made physical sense to ask about the change in the zero-point energy of the quantized electromagnetic field, brought about by a perfectly reflecting but infinitesimally thin material shell of radius R. If perfect reflection is taken to preclude any dependence on parameters of the material, then on dimensional grounds alone the answer must take the form $\Delta E = (\hbar c/R)C_B$, where the coefficient is a pure number. A value for it, $C_B \simeq 0.046$, was proposed in a landmark paper by Boyer (1968), inaugurating a notably varied sequence of ingenious contributions to mathematical physics, with generalizations to other types of field, to other boundary conditions satisfied by the field on the surfaces of the shell, and to spaces with other dimensionalities. All this is discussed in the recent review by Bordag *et al* (2001), who list many references.

On the other hand, if one considers thin macroscopic shells with realizable electromagnetic responses, and in particular the cohesive energies B due to their interaction with the electromagnetic field, then one notes at once that an expression such as ΔE can form only a very small part of B. This is clear from the fact that B is dominated by terms that are dependent on properties of the material, and are proportional to the amount of material present (i.e. to R^2 for thin shells). Although serious questions about the relation of ΔE to B have been visible[1] since the work of Davies (1972, section 4b) and Candelas (1982), satisfactory answers have been delayed by at least two sources of confusion, both related to the appearance of divergences in such calculations.

First, the initial approaches employed nondispersive response functions, in the hope that, eventually, allowance for dispersion would cure the divergences; whereas the theory for insulators already shows that it does not. Second, quantities divergent in the perfect-conductor limit tended to be discarded unexamined, as if they were wholly unphysical, by tacit analogy with the familiar procedures of renormalizable field theories; whereas it turns out that dropping them drops the true dominant components of B (Barton 2001a, 2002, Marachevsky 2001a, 2001b, referred to as B.I, B.II and M.I, M.II). Papers B.I and B.II argue that this amounts to confusing mathematical regularization with the physics of renormalization, and then in effect renormalizing inappropriately. That the analogy with standard field theory is false has been demonstrated from another point of view by Jaffe and his co-workers: cf appendix D for references and brief comment.

Here we aim to narrow such problems by determining B for a model inspired by the giant carbon molecule C_{60}. Specifically, we envisage a very thin spherical shell of arbitrary radius, structured much like a single hexagonal base plane from graphite having fully delocalized π but no σ electrons, with mean inter-electron spacing a. The electrons are treated as a continuous charged fluid, exactly as in standard hydrodynamic plasma models. (For 3D see e.g. Jackson (1975), section 10.8, and Fetter and Walecka (1971); for thin layers see Fetter (1973).) To find B for this system, we shall start from its total zero-point energy, subtract the zero-point energy of the Maxwell field in absence of the shell, and then subtract also the self-energy of its constituents, namely the self-energy of the same amount of material but at infinite dilution. As in all plasma theories, mathematically well-defined and physically interpretable energies emerge only on introducing a Debye-type cutoff on normal modes, reflecting the granularity of the true underlying electron gas. On the other hand no cutoff is needed on frequencies. Of course, it would be more realistic to treat the material as an assembly of discrete particles from the outset (see e.g. Renne (1971a, 1971b)): that however would eliminate precisely the

[1] So visible that, as should have been done long ago, we now abandon the quantity ΔE, which in fact is undefinable, and reassign the symbol C_B to what will eventually emerge as a small contribution to B, called the Boyer component.

features of interest to field theory, whose universal custom we follow by adhering to a strictly continuum model until the very last summation over normal modes.

We study plasmas rather than dispersive insulators (as in B.I, II and M.I, II) because plasmas are the least difficult to deal with of dispersive materials strongly coupled to the Maxwell field. It is of the essence that strong coupling can be achieved either intensively through large enough $x \equiv e^2/mc^2a$ at fixed R, or extensively through large enough $X \equiv R/a$ at fixed x. The first resembles the perfect-reflector limit adopted by many older approaches from the start, but requires impossibly high values of x; only the second can yield clues as to the energies of realistically reflecting macroscopic bodies. The price one pays, as compared to the perturbation theory which more or less suffices for insulators, is some quite delicate analysis, followed by algebra and integrations straightforward in principle but so extensive that in practice they have to be computerized[2]. What one gains is a rational classification plus a rough estimate of the main contributions to B, based on field theory, but manifestly akin to what one would expect from orthodox condensed-state physics. Remarkably, though the two strong-coupling limits differ about everything else, they supply the same closed-form expression $C_B = (3/64) - (9/4096)(\pi^2/8 - 1) + \cdots \simeq 0.0464$ for the first two nonzero terms of the *Boyer component*, defined (cf footnote 1) as the part of B proportional to $1/R$. Section 8.3 discusses what if anything can be learnt from comparing this with the value $0.0462\ldots$ given by recent calculations in the same spirit as Boyer's but using newer mathematics.

To summarize, the strength of our model is that it keeps close to the familiar physics of the electromagnetic field coupled to dispersive materials. It has the corresponding weakness that it is not renormalizable in the sense of orthodox field theory (cf appendix D), and not readily generalized to other dimensionalities, mainly because there is no well-founded and certainly no compelling generalization of the coupling mechanism. Another lack, probably less difficult to supply, is that the model ignores dissipation: its resonances for instance have only their natural widths. Meanwhile we cannot allow for Ohmic conduction, i.e. we cannot trade up from the plasma to what is often called the Drude model. To do that one would adapt the Huttner–Barnett theory with an appropriate reservoir of dissipative oscillators: for a clear acount of the principles see e.g. Huttner and Barnett (1992) and Barnett *et al* (1996).

The paper is organized as follows.

Section 2 defines the model, and identifies three scenarios, the only three we shall consider. They are specified in terms of the parameters x, X and $\mu \equiv 4\pi x X$ from (2.2). All three scenarios have $X \gg 1$. The *molecular scenario* has $x \ll 1$ and $\mu \ll 1$, as in C_{60}. At the other extreme, the *SF scenario* has $x \gg 1$, which would greatly simplify the calculations, but proves incompatible with the basic restriction that the plasma move nonrelativistically. The most interesting is the *macroscopic scenario*, with $x \ll 1$ but with X large enough to achieve $\mu \gg 1$. Cutoffs and subtractions are discussed in section 2.3. Section 3 determines B for the nonretarded version of the model (Coulomb forces only). The result is important because it does not rely on quantum field theory, but may nevertheless be compared with the limit $c \to \infty$ of several other expressions that do: agreement serves as a check that the prescription for subtracting self-energies is correct, and also against slips in the calculations.

The general theory in sections 4.1 and 4.2 then expresses the exact B in terms of the scattering phase shifts of the multipolar Maxwell fields. Section 4.3 describes and names the various types of contributions to B that will concern us, and sets out points of view important for structuring much of what follows. It also suggests a possible explanation for the fact

[2] The writer, for one, would never have completed sections 6 and 7 without MAPLE.

that in some other models and other dimensionalities the older methods[3] applied to perfect reflectors produce divergent expressions even for Boyer components. Sections 4.3 and 8.3 will suggest that such divergences might simply reflect the dependence of the true C_B on local coupling-strength parameters such as x, a very minor question as far as B itself is concerned.

Though the expressions from section 4 are exact, we can evaluate them only by exploiting the Debye (or uniform) expansions of the Bessel functions that figure in the phase shifts. Section 5 structures the consequent approximation to B, section 6 implements it to zero order, and section 7 to orders 1 and 2. For technical purposes, section 6 is the central one of the paper: it shows how the different scenarios require quite different procedures, and develops procedures for all three. The minimum mathematical input includes several auxiliary expansions that are relegated to appendices A and B; appendix C spells out some arguments about the macroscopic scenario, essential to understanding its mathematics but tedious except to activists. Remarkably, while order zero supplies all of the macroscopically dominant R^2-proportional component of B, it contributes nothing to the Boyer component, which cannot therefore be studied at all without going to order 1, nor with any confidence unless one looks also at order 2. Fortunately, the insights from section 6 allow section 7 to move fairly briskly.

Section 8.1 compares the contributions to B from different orders. This discussion is focussed on the table, which condenses a great deal of information, and needs to be considered quite carefully if one is to appreciate just how the Debye expansion works through to the end-results. Section 8.2 examines nonretarded limits, and section 8.3 discusses Boyer terms in more detail. Section 8.4 explains how B determines the pressure P, and anticipates some surprising properties of P that emerge in a following paper (Barton 2003a referred to as B.IV).

Finally, appendix D attempts a brief concordance between our theory of the plasma model and the theory of a renormalizable model considered by Jaffe and co-workers. The two theories generate very similar reservations about the assertions that older approaches base on the perfect-reflector limit, but there are technical differences stemming from different priorities in balancing physical verisimilitude against mathematical transparency.

Though the technicalities may appear heavy, especially in appendices A–C, they are no more than might just enable a sceptical reader to verify the conclusions in reasonable time, without having to start from scratch to identify crucial steps in the analysis, adapt analysis to calculation, and secure the numerous and somewhat offbeat expansions that the calculations require as input. Trying to re-orient the thrust of so much work done over thirty years and more, the writer believes that this is the minimum backing the arguments need to survive serious challenge.

2. Graphiteroles: the plasma model

2.1. Model parameters and the equations of motion

Our hydrodynamic plasma model consists of an infinitesimally thin spherical shell of continuous fluid, with radius R, mimicking a total number N of *delocalized charge carriers* (call them electrons for short), having charge[4] and mass e, m; plus an immobile, uniformly distributed, overall-neutralizing background charge. Sensible versions of the model relate n to some underlying lattice parameter or mean inter-electron distance a by

$$n \equiv N/4\pi R^2 \equiv 1/a^2. \tag{2.1}$$

The fluid displacement $\boldsymbol{\xi}$ and its canonical conjugate field $\boldsymbol{\Pi}$ are defined only on the shell, and

[3] See Milton (2003) and references there; for another critique see Jaffe and Graham *et al* as cited in appendix D.

[4] We use unrationalized Gaussian units. Thus the fine-structure constant reads $e^2/\hbar c \simeq 1/137$.

are purely tangential (suffix $\|$ on the Maxwell fields, but omitted as unnecessary on $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$). All equations of motion are linearized in $\boldsymbol{\xi}$ and in $\dot{\boldsymbol{\xi}}$ or $\boldsymbol{\Pi}$, and we assume that $|\dot{\boldsymbol{\xi}}| \ll c$.

The original version of this model, for an indefinitely extended flat sheet, was developed by Fetter (1973) with a view primarily to single base planes in graphite. It generalizes naturally to fullerenes, and in particular to the quasi-spherical shells constituting the giant carbon molecule[5] C_{60} and its even larger versions. In allusion to these prototypes, we call any such structure a *graphiterole*.

Our object is to determine the cohesive energy B of graphiteroles of arbitrary radius, i.e. their total zero-point energy less the combined self-energies of their constituent charge carriers at infinite dilution.

Define

$$r_0 \equiv e^2/mc^2, \qquad x \equiv r_0/a, \qquad X \equiv R/a, \qquad \mu \equiv 4\pi X x. \tag{2.2}$$

The intensive parameter x is the natural measure of the strength of the local coupling between the Maxwell field and the plasma sheet: it features the classical electron radius r_0 only through a coincidence of dimensionalities. By contrast, X is the natural measure of the size of the system. However, while the dominant parts of our end-results are best viewed as expansions in powers of x or of X, it is the product μ that turns out to govern the course of almost every nontrivial calculation.

For graphiteroles, a is of the order of a few Bohr radii, far larger than r_0:

$$a \sim a_B \equiv \hbar^2/me^2, \qquad x \sim (e^2/\hbar c)^2 \simeq (1/137)^2 \ll 1. \tag{2.3}$$

Evidently a/a_B would be a 2D analogue of the parameter usually called r_s in theories of the 3D electron gas (see e.g. Fetter and Walecka (1971)).

R is always treated as large, in the sense that $X \gg 1$, and we neglect all terms whose contributions to B would vanish faster than $1/R$. Section 3 gives some numerical estimates based on C_{60}. Here we anticipate only that it has $x_C \sim 10^{-5}$, and $\mu_C \sim 5 \times 10^{-4}$. Although μ becomes large for large enough R, the intensive parameter x_C is so minute that even a radius as large[6] as 1 micron produces only $\mu \sim 1$.

As section 2.3 explains in more detail, we impose a Debye-type cutoff on the angular momentum of the normal modes:

$$l \leqslant L, \qquad N = 4\pi X^2 \equiv \sum_{l=1}^{L}(2l+1) = L^2 + 2L \quad \Rightarrow \quad L = \sqrt{N+1} - 1 \simeq 2\pi^{1/2}X + \cdots, \tag{2.4}$$

where the last step follows because $X \gg 1$ entails $L \sim \mathcal{O}(X)$. The cutoff governs two important types of sum,

$$S_p(L) \equiv \sum_{l=1}^{L}(2l+1)^p, \qquad T_p(L) \equiv \sum_{l=1}^{L}(2l+1)^p \log(2l+1), \tag{2.5}$$

which we shall always approximate for large L, to accuracies dictated by the context.

[5] When improved by including σ electrons subject to restoring forces calibrated on graphite, our model (Barton and Eberlein 1991) roughly fits the measured frequencies and oscillator strengths of C_{60}. For more elaborate theories of C_{60} with similar aims see e.g. Ju *et al* (1993), and Vasváry (1996); for evidence from photoabsorption see e.g. Iglesias-Groth *et al* (2002); and from electron scattering, e.g. Gerchikov *et al* (1998).

[6] Of course, it is unrealistic to contemplate graphiteroles this big made of just a single layer of graphite. Multilayer shells admit nonzero radial displacements ξ_r of the plasma, with normal modes to match, requiring calculations more complicated by some orders of magnitude (see e.g. Fetter (1974)). As a poor man's alternative one might try to mimic a sheet with t layers by continuing to ignore radial oscillations, but increasing the surface density of charge carriers from n to tn. (However, readers might well be shocked by the value of t needed even to increase x to just 1.) Such adaptations are not considered in the present paper.

We shall consider three scenarios: *molecular*, with $x \ll 1$ and $\mu \ll 1$; *macroscopic, with $x \ll 1$ but $\mu \gg 1$, which is our main concern*; and *SF*, with $x \gg 1$, implying not only $\mu \gg 1$ but also $\mu/L \gg 1$. SF is both fanciful and inconsistent: for instance, it entails $|\dot{\boldsymbol{\xi}}| \sim c$; but it proves useful as an auxiliary to some calculations on the macroscopic scenario, and is formally reminiscent of the perfect-reflector limit considered in many older discussions of B. Its abnormalities are best discussed separately and in the simpler context of a flat plasma sheet (see a further paper, Barton (2003b), referred to as B.V). For the moment we anticipate only that it makes B negative, which might have been construed in its favour until one finds that, for the more or less realistic molecular and macroscopic scenarios, our model, like all purely plasma models, makes B positive. Truly cohesive energies, i.e. negative B, would emerge only on taking into account the interactions between the ions that make up the true neutralizing background: see e.g. Ashcroft and Mermin (1976).

For normal modes with all time-dependence in a common factor $\exp(-\mathrm{i}\omega t)$, Newton's second law yields

$$\boldsymbol{\xi} = -\frac{e}{m\omega^2}\mathbf{E}_\parallel, \tag{2.6}$$

disregarding the Lorentz force since $|\dot{\boldsymbol{\xi}}| \ll c$ by assumption; and Maxwell's equations read

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} - \mathrm{i}\omega\mathbf{B}/c = 0, \tag{2.7}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\delta(r-R)\sigma, \qquad \boldsymbol{\nabla}\times\mathbf{B} + \mathrm{i}\omega\mathbf{E}/c = 4\pi\delta(r-R)\mathbf{J}/c, \tag{2.8}$$

where the densities of surface current and surface charge are

$$\mathbf{J} = -\mathrm{i}\omega ne\boldsymbol{\xi}, \qquad \sigma = -ne\boldsymbol{\nabla}_\parallel\cdot\boldsymbol{\xi}. \tag{2.9}$$

To obtain the matching conditions on the fields, we integrate Maxwell's equations across the shell, and use (2.9), (2.6), which amounts to applying Gauss' law and Ampère's law. They yield

$$\mathrm{discont}(\mathbf{E}_\parallel) = \mathbf{0}, \qquad \mathrm{discont}(E_r) = \frac{(4\pi c^2 x/a)}{\omega^2}\boldsymbol{\nabla}_\parallel\cdot\mathbf{E}_\parallel, \tag{2.10}$$

$$\mathrm{discont}(B_r) = 0, \qquad \mathrm{discont}(\mathbf{B}_\parallel) = -\mathrm{i}\frac{(4\pi cx/a)}{\omega}\hat{\mathbf{r}}\times\mathbf{E}_\parallel. \tag{2.11}$$

2.2. The Hamiltonian

Finally we write down the Hamiltonian $\mathcal{H}$, choosing the Coulomb gauge. Then minimal coupling yields

$$\mathcal{H} = \mathcal{H}_{\mathrm{NR}} + \mathcal{H}_{\mathrm{int}} + \mathcal{H}_{\mathrm{rad}}, \tag{2.12}$$

where $\mathcal{H}_{\mathrm{rad}}$ is the Hamiltonian for the free Maxwell field;

$$\mathcal{H}_{\mathrm{NR}} = \int \mathrm{d}S\frac{\Pi^2}{2nm} + \frac{1}{2}\iint \mathrm{d}S\,\mathrm{d}S'\frac{\sigma(\mathbf{r})\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{2.13}$$

is the nonretarded Hamiltonian that one would have for the plasma sheet unaware of the Maxwell field; and the radiative coupling reads

$$\mathcal{H}_{\mathrm{int}} = \int \mathrm{d}S\left\{-\frac{e}{mc}\boldsymbol{\Pi}\cdot\mathbf{A}_\parallel + \frac{ne^2}{2mc^2}\mathbf{A}_\parallel^2\right\} \equiv \mathcal{H}_{\mathrm{int},1} + \mathcal{H}_{\mathrm{int},2}. \tag{2.14}$$

The $\int \mathrm{d}S \ldots$ are two-dimensional integrals running over the sheet: for spherical shells $\mathbf{r} = (R, \Omega)$ and $\int \mathrm{d}S \ldots \equiv R^2 \int \mathrm{d}\Omega \ldots$

In fact, apart from defining the system under discussion and elucidating the self-energy subtraction, the Hamiltonian will not be needed until B.IV, where the analysis of the pressure calls for the explicit connections between $\mathcal{H}_{\mathrm{NR}}$, its creation and annihilation operators, and the variables $\boldsymbol{\xi}$, $\boldsymbol{\Pi}$, plus the quantized Coulomb potential due to σ. But B will be calculated via the total zero-point energy (ZPE), which requires one only to solve the classical equations of motion so as to determine the frequencies of the normal modes. (One advantage of this is that we need not (yet) spend time on the several subtleties of the canonical commutation rules.) Paper B.V determines the cohesive energy for the analogous models for an unbounded medium in 3D and for an indefinitely extended flat sheet. These problems can be solved exactly, and afford both physical insight and technical confirmation of several results which for the spherical shell emerge only as approximations.

2.3. Self-energies, Debye cutoff and subtractions

To obtain B from the total ground-state energy of the coupled system, one drops, first, the ZPE of the Maxwell field in absence of the plasma; and second, the self-energy of the material evaluated at infinite dilution. To implement the second subtraction, one must introduce a cutoff by appeal to the granular nature of the true as opposed to the model fluid. That this is essential is known from the theory for insulators (B.I, B.II; M.I, M.II), where dispersion alone is insufficient to yield convergent (i.e. physically sensible) Casimir binding energies. Delocalized electrons present more of a challenge, because one must motivate reasonable and mutually compatible cutoffs on the wave-numbers of modes that, roughly speaking, are supported primarily by the degrees of freedom of the material (see section 3), and on photon modes supported primarily by the degrees of freedom of the Maxwell field. (This distinction is more readily visible for flat sheets, where true surface-bound modes survive even for finite c. By contrast, we shall see in section 4 that for spherical shells bound modes exist only in the nonretarded limit $c \to \infty$; for finite c they dissolve in the continuum of propagating modes, making the problem of cutoffs appear more delicate than it really is.)

Briefly, we propose *(i) to identify a conventional Debye cutoff*[7], $l \leqslant L$ *as in* (2.4), *by counting degrees of freedom for the fluid alone; and then (ii) to impose exactly the same cutoff also on photon modes*. Step (i) assigns to the fluid just one degree of freedom per particle, on the grounds that without the Maxwell field its motion would be purely longitudinal, in the sense that $\boldsymbol{\xi}$ would become expressible as the gradient of a scalar displacement-potential defined on the shell. (Admitting two modes per particle would have the same effect as simply doubling the model parameter n.)

In effect one is saying or pretending, with Debye[8] (1912), that matter modes forbidden by the cutoff do not exist at all. Exactly this is done in the next section. The reasons for step (ii) are more sophisticated, seeing that the Maxwell field off the plasma sheet certainly does have normal modes with arbitrarily high wave-numbers or angular momenta. But the modes

[7] Modes with given l have surface-parallel (tangential) wave-numbers of order $k_\parallel \sim l/R$, while a 2D gas with mean interparticle spacing a can support only waves with $ak_\parallel \lesssim 1$, entailing $l \lesssim R/a = X \sim L$. Thus our cutoff could equally well be interpreted as forbidding such impossible modes.

[8] The present paper relies heavily on three different ideas of Debye: the Debye (1909a) potentials for multipoles (section 4.1); the Debye (1909b) or uniform expansion of Bessel functions (appendix A.1); and the cutoff. The first two grow naturally from his thesis topic, the radiation pressure experienced by spheres. Surprisingly, there is a direct link also to the third idea: lacking Weyl's theorem (which dates from 1913), Debye chose to confirm the density of normal modes at high wave-number by determining it for the elastic vibrations of a sphere.

eliminated by the cutoff are deemed not to interact with the plasma, on the grounds that the collective excitations of the true granular material cannot respond to them.

Section 4 will show that subtracting the unwanted self-energies amounts to subtracting, under certain frequency integrals, the leading Born approximations from the pertinent multipolar phase shifts. This parallels, for the coupling between the Maxwell field and a plasma, the prescription developed by Jaffe and co-workers for scalar fields coupled to potentials (Graham *et al* 2001, 2002a, 2002b, 2003a). Unfortunately, the difference between the coupling mechanisms produces integrals whose structures are significantly different, so that we shall have to tackle ours from scratch.

In fact the Born subtractions are equivalent, as they should be, to dropping the energy $\langle 0|\mathcal{H}_{\mathrm{int},2}|0\rangle$ calculated with the familiar free-field expansion[9] of $\mathbf{A}$. We recall that this vacuum expectation-value measures the kinetic energy forced on the electrons by the zero-point oscillations of the electric field: see e.g. Barton (1989). The point is merely that the Born versions are convenient in the calculation, and other versions are not.

We stress that the Debye cutoff on surface-parallel wave-numbers is our only cutoff, and that it is rooted in the physics of real materials. By contrast, no artificial bounds are needed or imposed on frequency integrals: in virtue of the dispersion natural to the plasma, these all converge once one has made the two subtractions prescribed above.

3. The nonretarded model

This version of the plasma model is constructed by taking the nonretarded (NR) limit $c \to \infty$ at the outset. Then there is no Maxwell field to be quantized (hence no self-energies to be subtracted), and the Hamiltonian is just $\mathcal{H}_{\mathrm{NR}}$. As usual, the precise dimensionless criterion validating the limit becomes visible only *a posteriori*, from the dynamics it entails. Here the criterion will emerge as $x \ll 1$, appropriately to the molecular and the macroscopic scenarios of the general theory, whose leading terms indeed tally with that of the NR model. For further discussion of NR *limits* see section 8.2.

The normal modes are labelled l, m (where $-l \leqslant m \leqslant l$); the frequencies are independent of m in virtue of the spherical symmetry, and will be written as ω_l. They are found as in Barton and Eberlein (1991) by solving $\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}\neq\mathbf{0}) = 0$ subject to (2.10) and to $\mathbf{E}(\infty) = 0$, and read

$$\omega_l = \sqrt{\frac{4\pi ne^2}{mR}\frac{l(l+1)}{(2l+1)}} = \sqrt{\frac{Ne^2}{mR^3}\frac{l(l+1)}{(2l+1)}}, \qquad 1 \leqslant l \leqslant L. \tag{3.1}$$

The ground-state energy reads

$$B_{\mathrm{NR}} = \sum_{l=1}^{L}(2l+1)\frac{\hbar\omega_l}{2} = \hbar\sqrt{\frac{\pi ne^2}{mR}}\,S_{\mathrm{NR}}(L), \qquad S_{\mathrm{NR}}(L) \equiv \sum_{l=1}^{L}\sqrt{l(l+1)(2l+1)}. \tag{3.2}$$

Using the Abel–Plana formula, laboriously, one derives[10]

$$S_{\mathrm{NR}} = \sqrt{2}\left\{\frac{2}{5}L^{5/2} + L^{3/2} + \frac{7}{16}L^{1/2} - \frac{1}{32}L^{-1/2} + \mathcal{O}(L^{-3/2})\right\} + C_{\mathrm{NR}} \tag{3.3}$$

$$= \frac{16\pi^{5/4}}{5}X^{5/2} + \frac{3\pi^{1/4}}{8}X^{1/2} + C_{\mathrm{NR}} + \mathcal{O}(X^{-3/2}), \qquad C_{\mathrm{NR}} \simeq -0.127. \tag{3.4}$$

[9] To avoid over-correction, the vacuum expectation value must include only the normal modes admitted by the Debye cutoff.

[10] It is a welcome fact that the expression $\sqrt{2}\{(2/5)L^{5/2} + L^{3/2} + (7/16)L^{1/2}\} + C_{\mathrm{NR}}$ represents S_{NR} to 1% already for $L = 1, 2$, and to 0.1% for $L \geqslant 3$.

If (2.4) yields a non-integer L, one must fudge. For $L \gg 1$ it seems sensible to fudge by substituting from (2.4) directly into (3.3).

For later convenience we express $B_{\rm NR}$ in terms of X and (at this stage somewhat artificially) of x:

$$B_{\rm NR} = \frac{\hbar c}{a} x^{1/2} \left\{ \frac{16\pi^{7/4}}{5} X^2 + \frac{3\pi^{3/4}}{8} + \pi^{1/2} C_{\rm NR} X^{-1/2} + \mathcal{O}(X^{-2}) \right\}. \tag{3.5}$$

To illustrate orders of magnitude, we consider just the π-electrons in C_{60}, using $N = 60$ and the measured $R = 3.42$ Å. (Data from Taylor (1995).) Then the intensive parameters are

$$a_C = 1.57\,\text{Å} \quad \Rightarrow \quad x_C \equiv \frac{r_0}{a_C} \simeq 1.80 \times 10^{-5}, \qquad x_C^{1/2} \simeq 0.004\,24, \qquad x_C^{-1/2} \simeq 236. \tag{3.6}$$

They remain much the same for larger graphiteroles. The extensive parameters are $X_C \simeq 2.19$ and $\mu_C \simeq 4.94 \times 10^{-4}$. Then $\hbar\omega_1 \simeq 10$ eV, corresponding to a wavelength $2\pi c/\omega_1 \simeq 1200\,\text{Å} \gg R$. Taking $S_{\rm NR}$ from (3.4) with[11] $L = \sqrt{61} - 1 = 6.8$, we evaluate (3.2): this yields a ZPE per atom of $B_{\rm NR}/60 = +17$ eV, compared to the measured binding energy of -7.4 eV.

Since the normal modes of this model constitute a set of nonrelativistic oscillators, the virial theorem shows that on average half of $B_{\rm NR}$ is kinetic energy of the electrons, and thereby localized on the shell: hence only half is potential energy distributed over space, with a density $\langle 0|\mathbf{E}^2/8\pi|0\rangle$ determined in B.IV. The fact that not all of B need reside in the Maxwell field is a caveat to remember when assessing older approaches, which generally assume that it does.

Finally, (3.5) imposes a selfconsistency condition on the initial assumption in section 2.1 that the fluid moves nonrelativistically ($|\dot{\boldsymbol{\xi}}| \ll c$), with the mean kinetic energy $B_{\rm NR}/2$ far below the total rest energy. In view of the leading term from (3.5), and of $x = e^2/mc^2a$, the condition reads

$$\frac{1}{2}\frac{\hbar c}{a} x^{1/2} \frac{16\pi^{7/4}}{5} X^2 \ll 4\pi X^2 mc^2 \quad \Rightarrow \quad x^{3/2} \ll \left(\frac{5}{2\pi^{3/4}}\right)\frac{e^2}{\hbar c} \quad \Rightarrow \quad x \ll 0.0246. \tag{3.7}$$

Thus the NR model is admissible, as are the molecular and the macroscopic scenarios. For SF, the criterion $|\dot{\boldsymbol{\xi}}| \ll c$ takes a different form, because section 6.3 produces a different expression for B, call it $B_{\rm SF}$. But SF with its defining property $x \gg 1$ is still seen to be fictitious on observing that for $|B_{\rm SF}|/4\pi X^2 mc^2$ it yields an expression of order $x^2(\hbar c/e^2)$, which then is necessarily very large.

4. The general theory

4.1. Multipole modes and phase shifts

Following Bouwkamp and Casimir[12] (1954) (see also Jackson (1975), section 16.2) we represent the fields by Debye potentials ψ:

$$\mathbf{E}^{\rm TE}_{lm} = \mathrm{i}k\boldsymbol{\nabla} \times \left(\mathbf{r}\psi^{\rm TE}_{lm}\right), \qquad \mathbf{B}^{\rm TE}_{lm} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left(\mathbf{r}\psi^{\rm TE}_{lm}\right), \tag{4.1}$$

$$\mathbf{E}^{\rm TM}_{lm} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \left(\mathbf{r}\psi^{\rm TM}_{lm}\right), \qquad \mathbf{B}^{\rm TM}_{lm} = -\mathrm{i}k\boldsymbol{\nabla} \times \left(\mathbf{r}\psi^{\rm TM}_{lm}\right), \tag{4.2}$$

[11] Experimental evidence seems to exist only for plasmons with $l \leqslant 3$.

[12] Our TE and TM are their 'magnetic' and 'electric' multipoles respectively: see especially their table on p 547.

where the scalar functions $\psi_{lm}(\mathbf{r})$ solve the free Helmholtz equation off the shell, have the indicated angular dependence, and are regular at the origin:

$$k \equiv \omega/c; \qquad (\nabla^2 + k^2)\psi_{lm} = 0, \quad (r \neq R); \qquad \psi_{lm}(\mathbf{r}) = \phi_l(r)Y_{lm}(\Omega). \tag{4.3}$$

It will prove worth stressing that k is defined to be nonnegative.

To get a grip on the shifts in zero-point energies, one requires standing waves, i.e. real ϕ_l (for detailed discussion see e.g. Hagen (2000, 2002)). Hence[13]

$$\phi_l(r < R) = j_l(kr), \qquad \phi_l(r > R) = A_l\, j_l(kr + \delta_l), \tag{4.4}$$

$$\phi_l(kr \to \infty) \simeq (A_l/kr)\sin(kr + \delta_l - l\pi/2), \tag{4.5}$$

with A_l a constant (whose value is not required here);

$$(j_l(\xi),\, y_l(\xi)) = \sqrt{\frac{\pi}{2\xi}}(J_\nu(\xi),\, Y_\nu(\xi)), \qquad \nu \equiv l + 1/2, \tag{4.6}$$

$$h_l^{(1,2)}(\xi) = j_l(\xi) \pm \mathrm{i}y_l(\xi) = \sqrt{\frac{\pi}{2\xi}}H_\nu^{(1,2)}(\xi) \tag{4.7}$$

are the standard spherical Bessel functions. The Riccati–Bessel functions read

$$\left(\tilde{j}_l(\xi),\, \tilde{y}_l(\xi),\, \tilde{h}_l^{(1,2)}(\xi)\right) \equiv \xi\left(j_l(\xi),\, y_l(\xi),\, h_l^{(1,2)}(\xi)\right). \tag{4.8}$$

Equation (4.4) has already anticipated that Maxwell's equations plus the matching conditions admit no bound solutions, i.e. none that vanish as $r \to \infty$.

The phase shifts are governed by the matching conditions (2.10) and (2.11), which now reduce to

$$\mathrm{discont}(r\phi^{\mathrm{TE}}) = 0, \qquad \mathrm{discont}\frac{\partial}{\partial r}(r\phi^{\mathrm{TE}}) = \frac{4\pi x}{a}(r\phi^{\mathrm{TE}}), \tag{4.9}$$

$$\mathrm{discont}\frac{\partial}{\partial r}(r\phi^{\mathrm{TM}}) = 0, \qquad \mathrm{discont}(r\phi^{\mathrm{TM}}) = -\frac{4\pi x}{k^2 a}\frac{\partial}{\partial r}(r\phi^{\mathrm{TM}}). \tag{4.10}$$

Coincidentally, (4.9) makes our TE problem isomorphic to Schrödinger scattering by a repulsive potential[14] proportional to $\delta(R - r)$. Since the discontinuities must be finite, the limit $x \equiv r_0/a \to \infty$ enforces perfect reflection, through the boundary conditions $\phi^{\mathrm{TE}} = 0 = \partial(r\phi^{\mathrm{TM}})/\partial r$ at $r = R$.

From the matching conditions one eventually obtains the phase shifts

$$\tan\left(\delta_l^{\mathrm{TE}}(q)\right) = \frac{-\mu q j_l^2(q)}{1 - \mu q j_l(q) y_l(q)} = \frac{-(\mu/q)\tilde{j}_l^2(q)}{1 - (\mu/q)\tilde{j}_l(q)\tilde{y}_l(q)}, \qquad q \equiv kR, \tag{4.11}$$

$$\tan\left(\delta_l^{\mathrm{TM}}(q)\right) = \frac{-(\mu/q)\tilde{j}_l'^2(q)}{1 - (\mu/q)\tilde{j}_l'(q)\tilde{y}_l'(q)}. \tag{4.12}$$

As regards the δ_l at fixed q, it is $\mu \equiv 4\pi x X \to \infty$ that reproduces the familiar consequences of perfect reflection (see e.g. Jackson (1975), section 16.9). Remarkably, this can be envisaged as resulting either from strong coupling ($x \equiv r_0/a \to \infty$) as above, or from large radius ($X \equiv R/a \to \infty$), or formally from the dimensional limit $a \to 0$ which entails both.

[13] Our notation for Bessel functions is the same as in Watson (1944), the Bateman MS (ed Erdélyi (1953)), and Abramowitz and Stegun (1965).

[14] Gottfried (1966) treats the corresponding attractive potential.

In order to subtract the self-energies of the charge-carriers, we shall presently need the first Born approximations, i.e. the phase shifts to first order in μ:

$$\delta_l^{\mathrm{TE},B}(q) = -(\mu/q)\tilde{j}_l^2(q), \qquad \delta_l^{\mathrm{TM},B}(q) = -(\mu/q)\tilde{j}_l'^2(q). \tag{4.13}$$

The TE and TM amplitudes behave very differently as $q \to 0$:

$$\frac{\tilde{j}_l(q)\tilde{y}_l(q)}{q} \simeq -\frac{1}{(2l+1)}, \qquad \frac{\tilde{j}_l'(q)\tilde{y}_l'(q)}{q} \simeq \frac{l(l+1)}{(2l+1)}\frac{1}{q^2}. \tag{4.14}$$

Thus in (4.12), but not in (4.11), the denominator goes through zero close to $q = 0$ for any finite μ however small. We adopt the convention that all the phase shifts vanish as $q \to \infty$. Careful inspection then shows that

$$\delta_l^{\mathrm{TE}}(0) = 0, \qquad \delta_l^{\mathrm{TM}}(0) = -\pi. \tag{4.15}$$

The TM result reflects the fact that the exact theory dissolves the discrete NR bound mode in the TM continuum, which thereby contains (as section 4.2 will verify) one more state than it would in empty space[15]. It is the low-q behaviour of the TM phase shifts that links the general theory to the NR model: (4.14) shows that as q rises the denominator in (4.12) first vanishes when

$$1 = (\mu/q)\tilde{j}_l'(q)\tilde{y}_l'(q) \simeq \frac{l(l+1)}{(2l+1)}\frac{\mu}{q^2} \quad \Rightarrow \quad \omega = qc \simeq \omega_l,$$

with ω_l from (3.1).

4.2. Zero-point energy

Our strategy is as follows. (i) Impose the auxiliary boundary conditions that the ϕ_l vanish at $r = \mathcal{R} \gg R$; (ii) determine the *change* $\Delta\omega_n = c\Delta k_n$ induced by the phase shifts in the allowed values ω_n; (iii) for given (l, m) find the sum, call it Z_l, of the consequent *changes* in ZPE as $\mathcal{R} \to \infty$, converting $\sum_n$ into $\int \mathrm{d}k$ in (almost) standard fashion; (iv) subtract the appropriate counter-term, call it Z_l^*, to obtain $B_l = Z_l - Z_l^*$; (v) sum over (l, m) and over the polarizations. However, to ease the typography, we will wherever it is safe omit the polarization indices TE, TM, and the symbol $\sum_{\mathrm{pol}}$.

The asymptotics (4.5) entail

$$\cos[k_n\mathcal{R} + \delta_l(k_n)] = 0 \quad \Rightarrow \quad k_n\mathcal{R} + \delta_l(k_n) = (n + 1/2)\pi,$$
$$n = n_{\min}, (n_{\min}+1), (n_{\min}+2), \ldots.$$

Without the graphiterole, $n_{\min} = 0$. To determine $n_{\min}$ in presence of the graphiterole, we recall that k_n is nonnegative; and note that $n \to n_{\min}$ entails $\delta_n \to \delta(k = 0)$, whence (4.15) in turn implies $n_{\min}^{\mathrm{TE}} = 0$ and $n_{\min}^{\mathrm{TM}} = -1$, conformably to our earlier observations *à propos* of Levinson's theorem[16]. Finally, at very high n and thereby very high k_n, modes with and without the graphiterole correspond one to one, because $\delta_l(k_n \to \infty)$ vanishes. Hence

$$\Delta\omega_n = c\Delta k_n = -c\delta_l(k_n)/\mathcal{R}, \tag{4.16}$$

[15] In nonrelativistic potential scattering, Levinson's theorem reads $\delta_l(0) - \delta_l(\infty) = +\pi n_{lB}$, where n_{lB} is the number of bound states produced by the potential. The TE amplitudes satisfy the theorem, as they must in virtue of the isomorphy just noted. The TM amplitudes, unconstrained by any such coïncidence, conform to a kind of anti-Levinson theorem, as if n_B were equal to -1.

[16] These arguments are electromagnetic analogues of those used in mode-counting proofs of the theorem for potential scattering: see e.g. Barton (1985).

and the Euler–Maclaurin formula yields the precise version of the rule usually quoted in the somewhat loose form that $\sum_n \to \int \mathrm{d}n \to (\mathcal{R}/\pi)\int \mathrm{d}k$, namely[17]

$$\text{formally}: \quad Z_l = \sum_{n=n_{\min}}^{\infty} \frac{\hbar}{2}\Delta\omega_n = \frac{\hbar}{2}\left\{\frac{1}{2}\Delta\omega_{n_{\min}} - \frac{\mathcal{R}}{\pi}\int_{k_{n(\min)}}^{\infty} \mathrm{d}k \frac{c\delta_l(k)}{\mathcal{R}} + \cdots\right\}. \tag{4.17}$$

In fact we can simplify (4.17) by observing from (4.16) that the lower limit, the integrand, and the addend $\Delta\omega_{n_{\min}}/2$ are all of order $1/\mathcal{R}$; therefore, as $\mathcal{R} \to \infty$ we can drop the addend, and replace the lower limit by 0:

$$\text{formally}: \quad Z_l = -\frac{\hbar c}{2\pi}\int_0^{\infty} \mathrm{d}k\delta_l(k). \tag{4.18}$$

Next, one must subtract the self-energy of a fixed mass $4\pi X^2 m$ of fluid at infinite dilution, i.e. in the limit $\mu \to 0$ and $\delta_l \to 0$. Hence we obtain the subtrahend on replacing δ_l by the first Born approximation[18] δ_l^B from (4.13):

$$B_l = Z_l - Z_l^* = -\frac{\hbar c}{2\pi}\int_0^{\infty} \mathrm{d}k\left\{\delta_l(k) - \delta_l^B(k)\right\} = \frac{\hbar c}{2\pi R}F_l, \tag{4.19}$$

$$F_l \equiv -\int_0^{\infty} \mathrm{d}q\left\{\delta_l(q) - \delta_l^B(q)\right\}, \tag{4.20}$$

where the integrals converge in virtue of (4.11), (4.12).

With an eye to the Debye expansion in section 5, it proves convenient to connect F_l with B somewhat circuitously, defining the auxiliary variables

$$\alpha \equiv \frac{\mu}{2l+1} = \frac{\mu}{2\nu}, \qquad \nu \equiv l + \frac{1}{2}, \tag{4.21}$$

writing

$$B = \frac{\hbar c}{2\pi R}\sum_{l=1}^{L}(2l+1)F_l = \frac{\hbar c}{2\pi R}\sum_{l=1}^{L}(2l+1)^2\frac{F_l}{2\nu}, \tag{4.22}$$

and finally, by hindsight,

$$B = \frac{\hbar c}{4\pi R}H, \qquad H \equiv \sum_{l=1}^{L}\mathsf{h}_l, \qquad \mathsf{h}_l \equiv (2l+1)^2\frac{F_l}{\nu} = \frac{\mu^2}{\alpha^2}\frac{F_l}{\nu}. \tag{4.23}$$

For instance, the NR energy (3.5) emerges on replacing $H \to H_{\mathrm{NR}}$, where

$$H_{\mathrm{NR}} = x^{1/2}\left\{\frac{64\pi^{11/4}}{5}X^3 + \frac{3\pi^{7/4}}{2}X + 4\pi^{3/2}C_{\mathrm{NR}}X^{1/2} + \mathcal{O}(X^{-1})\right\}. \tag{4.24}$$

The first coefficient is enormous: $64\pi^{11/4}/5 \simeq 298.1$.

The sans-serif h_l in (4.23) should not be confused with the spherical Bessel functions $h_l^{(1.2)}$ (nor the function H with a Hamiltonian). Since the phase shifts are functions (beside q) only of μ, the F_l are functions of (μ, l) or of (μ, α), and H of (μ, L) or of (x, X). One reason for introducing the combinations just defined is that, in section 6.1, the leading Debye approximation $F_l^{(0)}/\nu$ to F_l/ν turns out to be a function only of α.

[17] The sum and the integral diverge, and must be understood as mere place-holders until subtraction of Z_l^* yields the well-defined equation (4.19).

[18] Jaffe and co-workers, as cited in section 2.3, arrive at formally the same prescription in an allied but technically somewhat different problem.

4.3. Some names and points of view

Components of B are called *extensive* if, at fixed a and x, they are proportional to R^2 (or equivalently to X^2 or to μ^2 or to N). Extensive components must dominate B for large enough R (i.e. formally in the thermodynamic limit where $N \to \infty$), and in our end-results they do, though some intermediate stages can look as if they might not. Components independent of R are called *shape-dependent* : their presence was first stressed by Candelas (1982). No special name seems to be current for components proportional to R.

If one were approaching B *de novo*, and from the point of view of condensed-state physics, it would be natural to drop terms that vanish as $R \to \infty$. But field-theorists are interested also in what section 1 has called the *Boyer component*, namely the term proportional to $1/R$, largely for the historical reason that in his classic calculation for perfect reflectors Boyer (1968) interpreted his expression for it as the *total* zero-point energy shift induced by the shell. Though this interpretation is wrong, we shall retain this component too, and analyse it on the same footing as the others. In H, the three types we have named are proportional respectively to X^3, X and X^0. But we do systematically drop components of H (rather than of B) that vanish as R (or X, or L) tend to infinity.

As regards the coefficient of any given power of X, one naturally expects it to depend on x, as those of X^3, X^2 and X in fact do. It proves convenient to have a label for terms that would diverge in the perfect-reflector limit $x \to \infty$, and we call them *nominally divergent*. There is nothing pejorative in the name, because the limit is purely hypothetical, and there is never any question of implementing it: the nominal divergence of an observable quantity is no more remarkable than the divergence of the ground-state energy of hydrogen as $e^2 \to \infty$.

However, in this respect the Boyer term does spring a surprise, at least according to the approximations we shall adopt. For $\mu \ll 1$ (e.g. in H_{NR}) we shall find no such term; and when for $\mu \gg 1$ it does turn up, it does not feature x at all. With an eye to other types of field and/or other dimensionalities, we shall call a Boyer term or coefficient *parametric* when the coefficient of X^0 does depend on material properties (in our case it might have featured x and/or $\log(X)$), and *nonparametric* when it does not. *A priori*, the former must be regarded as generic[19], i.e. as the rule, and the latter as an exception. Plasma shells in 3D with large μ are clearly exceptional. Presumably the Boyer coefficient vanishes with x and thereby with μ faster than any powers of μ (which are the best that our asymptotic approximations can deliver); whereas for $\mu \to \infty$ it becomes nonparametric in the sense that it is free of $\log(X)$ and tends to a finite limit, instead of increasing indefinitely with x or μ as do the coefficients of higher powers of X, and as physically speaking would be perfectly reasonable.

The introduction has already warned that the methods of this paper are too primitive to formulate let alone to answer corresponding questions about analogous systems having other dimensionalities. Only for an indefinitely extended right-circular[20] cylindrical shell in 3D does a related calculation, though not yet attempted, seem directly feasible. Neither can we prove that the Boyer component for large μ becomes nonparametric in the exact solution, nor even to all orders of the Debye expansion: we can merely proceed order by order.

However, if one accepts as a fact or as a working hypothesis that the Boyer component of H is indeed nonparametric, i.e. a pure number, then (4.23) shows that Boyer candidates in F_l/ν must be independent of α: otherwise they would feature the material constant x through

[19] For instance, B.I finds that thin and optically dilute insulating shells have a Boyer term proportional inter alia to $(\tau - 1)^2$, where τ is the optical thickness at zero frequency.

[20] Axis perpendicular to fixed circular cross-section.

their dependence on $\alpha = \mu/2\nu = 4\pi x X/2\nu$, which no subsequent summation over l can eliminate.

More general discussion of Boyer terms is postponed to section 8.3.

4.4. Jost functions and contour rotation

To put the integrals F_l into manageable form, it seems that one must express the phase shifts in terms of the multipolar S-matrix elements and their related Jost functions[21]. The connections are familiar from ordinary scattering theory, and this section draws freely on well-established structures and theorems obtainable say from Goldberger and Watson (1964) and from Newton[22] (1982). Our task is the easier because the requisite analyticity, asymptotic and convergence properties are visible directly from (4.4), (4.11), (4.12), so that they do not need to be established expensively from the underlying differential equations, as in potential theory.

The Jost functions, $f_l(q)$ and $f_l(-q)$, are defined so that $f_l(q)$ is analytic in the lower half-plane $\mathrm{Im}\, q < 0$ (it has no poles because there are no bound modes), and so that the S matrix is given by

$$\mathcal{S}_l(q) \equiv \exp(2\mathrm{i}\delta_l) = \frac{f_l(q)}{f_l(-q)} \quad \Rightarrow \quad \delta_l = \frac{1}{2\mathrm{i}}\{\log[f_l(q)] - \log[f_l(-q)]\}. \tag{4.25}$$

In our model, some manipulations using (4.7) and (4.11) or (4.12) then identify

$$\mathcal{S}_l^{\mathrm{TE}} = \frac{1 - \mathrm{i}\mu q j_l(q) h_l^{(2)}(q)}{1 + \mathrm{i}\mu q j_l(q) h_l^{(1)}(q)} \quad \Rightarrow \quad f_l^{\mathrm{TE}}(\pm q) = 1 \mp \mathrm{i}\mu q j_l(q) h_l^{(2,1)}(q), \tag{4.26}$$

and similarly

$$f_l^{\mathrm{TM}}(\pm q) = 1 \mp \mathrm{i}\frac{\mu}{q}\tilde{j}_l'(q)\tilde{h}_l^{(2,1)'}(q). \tag{4.27}$$

Next, we substitute δ_l from (4.25) into (4.20); and in the terms with $h_l^{(2)}$, $h_l^{(1)}$ try to rotate the integration contour to the negative and the positive imaginary axis respectively. (The integrals F_l converge at infinity because the $|g_l(y \to \infty|$ vanish like $1/2y$.) For TE there are no snags, and one finds

$$F_l^{\mathrm{TE}} = \int_0^\infty \mathrm{d}y \Phi_l^{\mathrm{TE}}(y), \qquad \Phi_l^{\mathrm{TE}} \equiv \log\left[1 + \mu g_l^{\mathrm{TE}}(y)\right] - \mu g_l^{\mathrm{TE}}(y), \qquad g_l^{\mathrm{TE}}(y) \equiv I_\nu(y) K_\nu(y). \tag{4.28}$$

But for TM one meets the difficulty, implicit in (4.14), that the integral over either of the two logarithms in (4.25), if taken on its own, diverges at $q = 0$. This makes it essential that the element next to the origin be common to both contours. Accordingly, for the term with $h_l^{(2)}$, we adopt a contour running from the origin to $q = \eta$ along the real axis, then along the quarter-circle from η to $-\mathrm{i}\eta$, and finally along the imaginary axis from $-\mathrm{i}\eta$ to $-\mathrm{i}\infty$; make the corresponding changes in the term with $h_l^{(1)}$; and finally take the limit $\eta \to 0$. The end-result reads

$$F_l^{\mathrm{TM}} = \int_0^\infty \mathrm{d}y \Phi_l^{\mathrm{TM}}(y), \qquad \Phi_l^{\mathrm{TM}} \equiv \log\left[1 + \mu g_l^{\mathrm{TM}}(y)\right] - \mu g_l^{\mathrm{TM}}(y) + \mu \Lambda_l / y^2, \tag{4.29}$$

where

$$g_l^{\mathrm{TM}}(y) \equiv -\frac{1}{y}\frac{\mathrm{d}(\sqrt{y} I_\nu(y))}{\mathrm{d}y}\frac{\mathrm{d}(\sqrt{y} K_\nu(y))}{\mathrm{d}y} = -\left\{ I_\nu' K_\nu' + \frac{(I_\nu' K_\nu + I_\nu K_\nu')}{2y} + \frac{I_\nu K_\nu}{4y^2} \right\}, \tag{4.30}$$

[21] The idea has been applied widely and quite variously. For references see e.g. the review by Bordag *et al* (2001).

[22] Caution: Newton's choices of sign connected with the Jost functions are peculiarly his own.

and

$$\Lambda_l \equiv \lim_{y\to 0}\left(y^2 g_l^{\mathrm{TM}}(y)\right) = \frac{l(l+1)}{(2l+1)} \quad \Leftrightarrow \quad g_l^{\mathrm{TM}}(y\to 0) \simeq \frac{\Lambda_l}{y^2}. \tag{4.31}$$

This is the bridge to the discrete nonretarded eigenfrequencies ω_l from (3.1), as already noted at the end of section 4.1, and thereby to B_{NR}, as spelled out in section 4.5.

4.5. Small μ

For TM, the double pole of $g_l^{\mathrm{TM}}(y)$ at the origin entails that for small μ the integral F_l^{TM} is dominated by small y. Hence one simply sets $\mu g_l^{\mathrm{TM}} \simeq \mu\Lambda_l/y^2$ and scales via $y = \xi\sqrt{\mu\Lambda}$:

$$F_l^{\mathrm{TM}} \simeq \sqrt{\mu\Lambda_l}\int_0^\infty \mathrm{d}\xi\, \log(1+1/\xi^2) = \pi\sqrt{\mu\Lambda_l}. \tag{4.32}$$

Substituting for μ and for Λ_l, and then into (4.22), it is easy to verify that (4.32) reduces B the nonretarded result[23] B_{NR} given by (3.2).

To clinch the connection we check that, for small μ, F_l^{TE} is much smaller than F_l^{TM}. One expands the integrand of (4.28) by powers of μ:

$$F_l^{\mathrm{TE}} = -\frac{\mu^2}{2}\int_0^\infty \mathrm{d}y\{I_\nu(y)K_\nu(y)\}^2 + \mathcal{O}(\mu^3), \tag{4.33}$$

and estimates the integral by using just the first term of the Debye expansion from appendix A. This yields $I_\nu(y)K_\nu(y) \simeq 1/2\sqrt{\nu^2+y^2}$, whence $\int_0^\infty \mathrm{d}y(I_\nu K_\nu)^2 \simeq \pi/4$ and $F_l^{\mathrm{TE}} \simeq -\pi\mu^2/8(2l+1)$. Thus $\left|F_l^{\mathrm{TE}}\big/F_l^{\mathrm{TM}}\right| \sim (\mu/l)^{3/2} \ll 1$, as expected.

5. The Debye expansion: structure

Since the sum in (4.23) would diverge without a cutoff, it is dominated by contributions from large $l \sim L$, and the integrals F_l with such l are themselves dominated by regions where y is comparable to l. This makes it natural to use the Debye (also called the uniform) expansions of $I_\nu(y)$ and $K_\nu(y)$, which are effectively expansions in powers of $1/\nu$. More specifically one defines

$$z \equiv \frac{y}{\nu} = \frac{2y}{2l+1}, \qquad t \equiv \frac{1}{\sqrt{z^2+1}}, \tag{5.1}$$

and treats z as of order unity. We shall implement the expansion to orders 0, 1 and 2. The order is indicated as a superscript, e.g. $F_l^{(0)}$ and $H^{(0)}$ in zero order, and so on.

Details are relegated to appendix A. In outline, we proceed as follows. (i) The functions g_l, which are central to the integrands Φ_l, are put into the form

$$g_l^{\mathrm{TE}} = G_l^{\mathrm{TE}}\left[1 + \frac{c_1^{\mathrm{TE}}}{\nu^2} + \frac{c_2^{\mathrm{TE}}}{\nu^4} + \cdots\right], \qquad G_l^{\mathrm{TE}} \equiv \frac{t}{2\nu}, \tag{5.2}$$

$$g_l^{\mathrm{TM}} = G_l^{\mathrm{TM}}\left[1 + \frac{c_1^{\mathrm{TM}}}{\nu^2} + \frac{c_2^{\mathrm{TM}}}{\nu^4} + \cdots\right], \qquad G_l^{\mathrm{TM}} \equiv \frac{1}{2\nu t}\frac{1}{z^2}, \tag{5.3}$$

where the $c_j^{\mathrm{TE,TM}}$ are polynomials in t (we suppress the further label l that they ought to carry). The constant Λ in Φ_l^{TM} is expanded correspondingly. (ii) The integrands Φ_l are

[23] It follows that the nonretarded limit of the Debye expansion will reproduce B_{NR} if and only if it reproduces Λ. Appendix A.1 shows that orders 0 and 1 combined achieve just that.

themselves expanded in powers of $1/\nu^2$. (iii) By hindsight, we define $\alpha = \mu/2\nu$ as in (4.21), systematically replace $1/\nu^2 \to (2\alpha/\mu)^2$, and write

$$\Phi_l = \Phi_l^{(0)} + \frac{1}{\mu^2}\Phi_l^{(1)} + \frac{1}{\mu^4}\Phi_l^{(2)} + \cdots. \tag{5.4}$$

For instance, $\Phi_l^{\mathrm{TE}(0)} = \left\{\log\left[1+\mu G_l^{\mathrm{TE}}\right] - \mu G_l^{\mathrm{TE}}\right\}$ is got from Φ_l^{TE} simply by replacing $g_l^{\mathrm{TE}} \to G_l^{\mathrm{TE}}$. More generally, the $\Phi_l^{(n)}$ now feature only α, t, and for TM also z. Finally (iv), in the integrals (4.28), (4.29) replace $\int \mathrm{d}y \ldots \to \nu \int \mathrm{d}z \ldots$, and rearrange them to read

$$\frac{1}{\nu}\left\{F_l^{(0)} + F_l^{(1)} + F_l^{(2)} + \cdots\right\} = \int_0^\infty \mathrm{d}z \left\{\Phi_l^{(0)} + \frac{1}{\mu^2}\Phi_l^{(1)} + \frac{1}{\mu^4}\Phi_l^{(2)} + \cdots\right\}. \tag{5.5}$$

Remarkably, the format on the right remains equally convenient for μ large or small. The $c_j^{\mathrm{TE,TM}}$ and the $\Phi_l^{(n)}$ are given in appendix A.1. The combinations $F_l^{(n)}\big/\nu$ will be kept intact, to ease the bookkeeping.

6. The Debye expansion: order zero

The main object is to find $H^{(0)}$ for use in (4.23), i.e. to find H to zero order. Subsection 6.1 determines the $F_l^{(0)}\big/\nu$, and their asymptotics for small and for large α. The other subsections determine $H^{(0)}$ for the three scenarios in turn, relying heavily on the sums S_p and T_p from appendix B. The molecular and SF scenarios involve only the asymptotics of the $F_l^{(0)}\big/\nu$, and could be viewed as preparatory for tackling the macroscopic scenario, which involves all values of α. In fact, although the macroscopic is the most interesting scenario, and its analysis perhaps the most novel, the elaboration it requires creates a dilemma: as probably the least awkward choice, much of the detailed argument is relegated to appendix C, with conclusions that subsection 6.4 will be content merely to quote.

A secondary object is to establish the main ideas and procedures thoroughly enough for section 7 to implement orders 1 and 2 without excessive repetition. For instance, one learns to recognize the special role played in the molecular and the macroscopic scenarios by components of H that are functions only of μ, which we call *jokers*.

It will turn out that order zero supplies the entire extensive component of B, but nothing towards the Boyer component. The contributions from the different orders are compared systematically in section 8.1.

6.1. The summands $F_l^{(0)}$

To evaluate $F_l^{\mathrm{TE}(0)}$ and $F_l^{\mathrm{TM}(0)}$ one rationalizes the integrand by changing variables from z to $x = \sqrt{1+z^2} - z$. Eventually one finds[24]

$$\frac{1}{\nu}F_l^{\mathrm{TE}(0)} = \alpha - \frac{\pi}{2} + 2\sqrt{1-\alpha^2}\tan^{-1}\left[\sqrt{\left(\frac{1-\alpha}{1+\alpha}\right)}\right], \qquad (\alpha < 1); \tag{6.1}$$

when $\alpha > 1$, one must replace $\sqrt{1-\alpha^2}\tan^{-1}[\sqrt{(\cdots)}] \to -\sqrt{-1+\alpha^2}\tanh^{-1}[\sqrt{-(\cdots)}]$.

[24] All logs, arctans and arctanhs in this paper are on their principal branches.

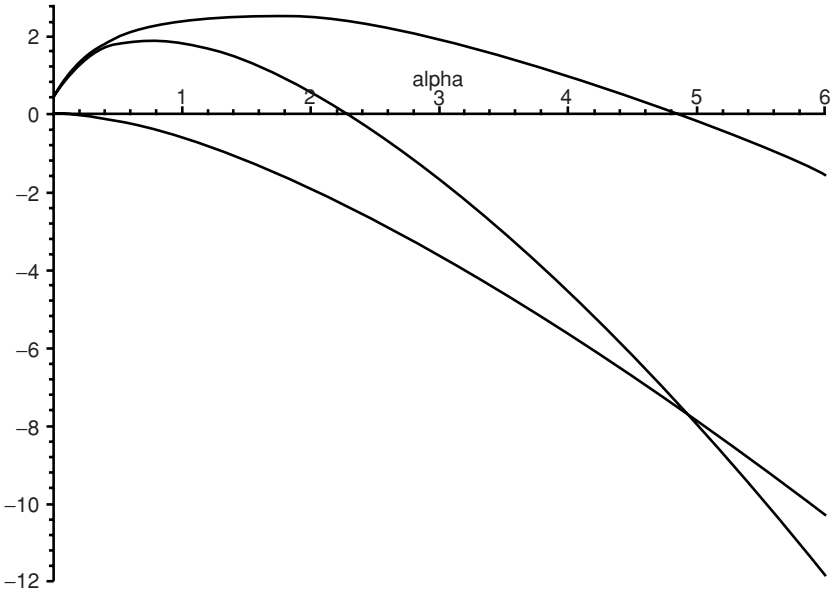

Figure 1. Zero-order Debye approximation to the summands F_l/ν in (4.22), (4.23). Top line: $F_l^{\mathrm{TM}(0)}(\alpha)/\nu$, equation (6.2); all-negative line: $F_l^{\mathrm{TE}(0)}(\alpha)/\nu$, equation (6.1), continued appropriately to $\alpha > 1$; and their sum $F_l^{(0)}/\nu = (F_l^{\mathrm{TE}(0)} + F_l^{\mathrm{TM}(0)})/\nu$.

Similarly, but for any α,

$$\frac{1}{\nu}F_l^{\mathrm{TM}(0)} = \left\{2\alpha + \sqrt{2}\sqrt{\alpha\sqrt{\alpha^2+4}-\alpha^2}\,\tan^{-1}\left[\frac{\sqrt{\alpha\sqrt{\alpha^2+4}+\alpha^2}\left(\sqrt{\alpha^2+4}-\alpha+2\right)}{2\sqrt{2}\alpha}\right]\right.$$
$$\left. - \sqrt{2}\sqrt{\alpha\sqrt{\alpha^2+4}+\alpha^2}\,\tanh^{-1}\left[\frac{\sqrt{\alpha\sqrt{\alpha^2+4}-\alpha^2}\left(\sqrt{\alpha^2+4}+\alpha-2\right)}{2\sqrt{2}\alpha}\right]\right\}. \tag{6.2}$$

The functions $F_l^{\mathrm{TE,TM}(0)}\big/\nu$ and $F_l^{(0)}\big/\nu = \left(F_l^{\mathrm{TE}(0)} + F_l^{\mathrm{TM}(0)}\right)\big/\nu$ are shown in figure 1: attention is drawn to the signs. In particular we shall need the asymptotics of $F_l^{(0)}\big/\nu$:

$$\alpha \to 0: \quad \frac{1}{\nu}F_l^{(0)} = \left\{\pi\alpha^{1/2} - \frac{\pi}{4}\alpha^{3/2} - \frac{\pi}{4}\alpha^2 + \frac{\pi}{32}\alpha^{5/2} + \frac{4}{15}\alpha^3 + \frac{\pi}{128}\alpha^{7/2}\right.$$
$$\left. - \frac{\pi}{16}\alpha^4 - \frac{5\pi}{2048}\alpha^{9/2} + \frac{44}{315}\alpha^5 - \frac{7\pi}{8192}\alpha^{11/2} + \mathcal{O}(\alpha^6)\right\}, \tag{6.3}$$

$$\alpha \to \infty: \quad \frac{1}{\nu}F_l^{(0)} = \left\{[-2\log(2\alpha)+3]\alpha + \frac{1}{2}\frac{1}{\alpha} - \frac{\pi}{4}\frac{1}{\alpha^2} + \left[\frac{3}{4}\log(2\alpha) - \frac{31}{48}\right]\frac{1}{\alpha^3}\right.$$
$$\left. + \frac{7\pi}{16}\frac{1}{\alpha^4} + \left[-\frac{5}{4}\log(2\alpha) + \frac{29}{20}\right]\frac{1}{\alpha^5} + \mathcal{O}\left[\frac{1}{\alpha^6}\right]\right\}. \tag{6.4}$$

In the low-α series, all fractional powers of α are TM, and all even powers are TE. By contrast, in the high-α series, both TE and TM contribute to the leading order $\alpha\log(\alpha)$.

Further, it is noteworthy that (6.4) jumps from α to $1/\alpha$, lacking a term independent of α, the only type capable of generating nonparametric Boyer components, as explained in

section 4.3. This stems from a cancellation between contributions $\mp\pi/2$ from TE and TM respectively. In fact $F_l^{(0)}/\nu$ will produce no Boyer components of either kind: for these one must therefore look to higher orders.

It remains to evaluate the sums over l featured in (4.23). This must be done very differently in our three scenarios, because it depends critically on the magnitude of μ. But all end-results will be arranged by descending powers of $X \gg 1$.

6.2. Molecular scenario

Here one has $\mu \ll 1 \quad \Rightarrow \quad \alpha = \mu/(2l+1) \ll 1$. Then (4.23) with the low-α series (6.3) for $F_l^{(0)}/\nu$ yields

$$H^{(0)} = \left\{ \pi\mu^{1/2} S_{3/2} - \frac{\pi}{4}\mu^{3/2} S_{1/2} - \frac{\pi}{4}\mu^2 S_0 + \frac{\pi}{32}\mu^{5/2} S_{-1/2} + \frac{4}{15}\mu^3 S_{-1} + \frac{\pi}{128}\mu^{7/2} S_{-3/2} + \cdots \right\}, \tag{6.5}$$

featuring the sums S_p defined in (2.5) and given by appendix B. The expansion has been carried as far as $\mu^{7/2} S_{-3/2}$ not from any preconceived desire for accuracy to order $\mu^{7/2}$, but because it is the first term having a power of μ higher than 3, and a sum that converges as $L \to \infty$, both being features that will help to elucidate the structure of H more generally.

In this scenario the mathematician would probably start by viewing $H^{(0)}$ as a series in rising powers of the small parameter x, with coefficients that feature X and L only through the combinations $\mu^{p-1} S_p$ or equivalently $X^{p-1} S_p$. We approximate them by expanding the S_p by falling powers of X, as in appendix B.3; and by systematically dropping from the $X^{p-1} S_p$ all terms that would vanish as X tends to infinity. The boxes surround terms whose contributions to H are what we have called *jokers*, defined as functions only of xX, i.e. only of μ. We shall have to watch them carefully when we apply the results of the molecular to the macroscopic scenario. One finds

$$H^{(0)} \simeq \sum_n C_n X^n + \mathcal{O}(X^{-1}), \tag{6.6}$$

$$C_{7/2} = \boxed{\pi^{9/2}\{-1 + \zeta(3/2)(1 - 2^{-3/2})\} x^{7/2}},$$

$$C_3 = \left\{ \frac{64\pi^{11/4}}{5} x^{1/2} - \frac{16\pi^{13/4}}{3} x^{3/2} - 8\pi^{7/2} x^2 + 2\pi^{15/4} x^{5/2} + (128\pi^3/15)\log(X) x^3 + \boxed{(128\pi^3/15)\{-2 + \gamma + \log(8\pi^{1/2})\} x^3} - \frac{\pi^{17/4}}{2} x^{7/2} \right\},$$

$$C_{5/2} = \boxed{\pi^{7/2}\{-1 + \zeta(1/2)(1 - 2^{-1/2})\} x^{5/2}}, \qquad C_2 = \boxed{\{4\pi^3 x^2\}},$$

$$C_{3/2} = \boxed{2\pi^{5/2}\{1 + \zeta(-1/2)(2^{1/2} - 1\})x^{3/2}},$$

$$C_1 = \frac{7\pi^{7/4}}{2} x^{1/2} - \frac{23\pi^{9/4}}{24} x^{3/2} - \pi^{5/2} x^2 + \frac{25\pi^{11/4}}{192} x^{5/2} + \frac{52\pi^2}{45} x^3 + \frac{9\pi^{13/4}}{256} x^{7/2},$$

$$C_{1/2} = \boxed{2\pi^{3/2}\{-1 + \zeta(-3/2)(1 - 2^{3/2})\} x^{1/2}}.$$

Several points are worth noting.

- All fractional powers of X are jokers. But the converse is not true.
- If (6.6) remained valid for macroscopic and therefore arbitrarily large X, then the joker of order[25] $(xX)^{7/2}$ would eventually dominate, making B nonextensive not only logarithmically but in good earnest. But in fact all jokers here, being proportional to μ^p with $p \geqslant 1/2$, are minute compared to the leading term:

$$(p \geqslant 1/2, \mu \ll 1, X \gg 1) \quad \Rightarrow \quad \frac{\mu^p}{x^{1/2}X^3} \sim \frac{\mu^{p-1/2}}{X^{5/2}} \ll 1.$$

- Logarithms are irrelevant to orders of magnitude, but deserve attention in some other respects. For instance, the $\log(X)$ multiplying C_3X^3 means that this, the leading term, is not properly extensive.
- There is no Boyer term, i.e. none proportional to X^0.
- The nonretarded limit is discussed in section 8.2.

6.3. SF scenario

Here x, X, μ, and all $\alpha = \mu/(2l+1)$ are large, with L and X linked by (2.4). Then the high-α series (6.4) for $F_l^{(0)}\big/\nu$ leads to

$$H^{(0)} = \left\{2\mu[(-2\log(2\mu)+3)S_1 + 2T_1] + \frac{1}{2\mu}S_3 - \frac{\pi}{4\mu^2}S_4 + \frac{1}{\mu^3}\left[\left(\frac{3}{4}\log(2\mu) - \frac{31}{48}\right)S_5 - \frac{3}{4}T_5\right] + \mathcal{O}\left(\frac{1}{\mu^4}S_6\right)\right\}. \tag{6.7}$$

Using appendix B, we expand the S_p and T_p by inverse powers of L and then of x, dropping terms whose contribution to $H^{(0)}$ would vanish as L and/or x tend to infinity: this turns (6.7) into an expansion effectively by descending powers of x, give or take some logarithms. Reorganizing by powers of X, one obtains

$$H^{(0)} \simeq D_3X^3 + D_1X + \mathcal{O}(X^{-1}), \tag{6.8}$$

$$D_3 = -32\pi^2[\log(2x\pi^{1/2}) - 1]x + 4\pi\frac{1}{x} - \frac{8\pi^{3/2}}{5}\frac{1}{x^2} + \frac{[36\log(2x\pi^{1/2}) - 25]}{9}\frac{1}{x^3},$$

$$D_1 = \left\{\pi\left[\frac{22}{3}\log(X\pi^{1/2}) + 16\log(2) - \frac{2}{3} + 8\pi\zeta'(-1)\right]x + \frac{3}{2}\frac{1}{x} - \frac{2\pi^{1/2}}{3}\frac{1}{x^2} + \frac{[252\log(2x\pi^{1/2}) - 181]}{144\pi}\frac{1}{x^3}\right\}.$$

Again there is no Boyer component.

Note that the leading term D_3X^3 in (6.8) is properly extensive, because it features $\log(x)$ rather than $\log(X)$ as C_3X^3 does in the molecular scenario. Since x is large and $\log(x)$ positive, D_3X^3 is negative, which as section 2.1 has already stated is unrealistic for a plasma model. Technically the sign reflects the negative sign of the $F_l^{(0)}\big/\nu$ at high α, visible from figure 1. Paper B.V finds similar sign reversals for an unbounded 3D plasma and for an indefinitely extended flat sheet. They are merely poor attempts to mimic certain extreme-relativistic effects: they must not be taken seriously, the less so because, as already anticipated at the end of section 3, our SF scenario is not internally consistent.

[25] And other jokers with higher powers of μ already dropped from (6.5).

6.4. Macroscopic scenario

6.4.1. Structure. This scenario applies when μ is large enough to make $\alpha \gg 1$ for $1 \leqslant l \leqslant L_1$, with some $L_1 \gg 1$, while nevertheless $\alpha \ll 1$ for $L_2 \leqslant l \leqslant L$, with some $L_2 \gg L_1$. It is perhaps the most interesting of the three, since it covers shells with roughly realistic x and macroscopic radii. It is also the most awkward, because, while one may use (6.4) for $l \leqslant L_1$ and (6.3) for $L_2 \leqslant l \leqslant L$, one must use the full expressions (6.1), (6.2) for $L_1 < l < L_2$, where α is neither large nor small.

In order to take maximum advantage of the molecular and SF sums already found, we first define a new symbol A, related to L as α is to l:

$$A \equiv \frac{\mu}{2L+1} = \frac{4\pi x X}{2\sqrt{4\pi X^2+1}-1}, \qquad L = \frac{\mu}{2A} - \frac{1}{2}; \tag{6.9}$$

the option of expressing A as a function of X rather than of L will prove crucial. Then we split the sum $H^{(0)}$ into three, at L_1 and L_2 such that

$$A_1 \equiv A(L_1) \gg 1, \qquad 1 \ll L_1 \ll L_2 \ll L, \qquad A_2 \equiv A(L_2) \ll 1,$$

and choose

$$A_1 = 10, \qquad A_2 = 0.1. \tag{6.10}$$

Hindsight suggests the grouping

$$H^{(0)} = \sum_{l=1}^{L_1} \mathsf{h}_l + \left[-\mathsf{h}_{L_1} + \sum_{l=L_1}^{L_2} \mathsf{h}_l \right] + \sum_{l=L_2+1}^{L} \mathsf{h}_l \equiv H^{(0)}_{\text{low}} + \left[H^{(0)}_{\text{mid}} \right] + H^{(0)}_{\text{high}}. \tag{6.11}$$

The label *low* refers to low[26] L and high α, and vice versa for the label *high*. Evidently $H^{(0)}_{\text{low}}$ and $H^{(0)}_{\text{mid}}$ are functions of μ alone; only $H^{(0)}_{\text{high}}$ can, through L, depend separately on x and X.

The SF scenario all but delivers $H^{(0)}_{\text{low}}$; $H^{(0)}_{\text{mid}}$ is evaluated numerically using the Euler–Maclaurin formula; and the molecular scenario helps to determine $H^{(0)}_{\text{high}}$, but only after some quite delicate rearrangements. Much of the detail concerning $H^{(0)}_{\text{high}}$ will be relegated to appendix C.

6.4.2. The sum $H^{(0)}_{\text{low}}$. Evidently $H^{(0)}_{\text{low}}$ follows from the SF result (6.7) on replacing $L \to L_1$; but one must remember that now we have $x \ll 1$ instead of $x \gg 1$. The most convenient procedure (i) expands the S_p and T_p for large L_1, exactly as before; (ii) expresses L_1 in terms of μ and A_1 according to (6.9); and finally (iii) substitutes $A_1 = 10$.

We choose to expand the result by inverse powers of μ, and then to expand the coefficients by inverse powers of A_1. Some reflection shows that $H^{(0)}_{\text{low}}$ has (in our case lacks) exactly the same Boyer component as the SF scenario, because B_B comes from multipoles with l well below L_1, and *a fortiori* well below L, which was all that signified in SF.

Omitting the intermediate steps, we merely quote

$$H^{(0)}_{\text{low}} = u_3\mu^3 + u_2\mu^2 + u_1\mu + \mathcal{O}(\mu^{-1}), \tag{6.12}$$

$$u_3 \simeq -\frac{[\log(2A_1)-1]}{2A_1^2} + \frac{1}{16A_1^4} - \frac{\pi}{40A_1^5} + \frac{[\log(2A_1)-25/36]}{16A_1^6},$$
$$u_2 \simeq -\frac{[\log(2A_1)-3/2]}{A_1} + \frac{1}{4A_1^3} - \frac{\pi}{8A_1^4} + \frac{[3\log(2A_1)-31/12]}{8A_1^6},$$

[26] Or rather to *relatively* low L, because even L_1 is far above unity.

$$u_1 \simeq \left[\frac{11}{6}\log(2\mu) - \frac{1}{3}\log(A_1) - \frac{23}{12} + 2\zeta'(-1)\right] + \frac{1}{4A_1^2} - \frac{\pi}{6A_1^3} + \frac{[5\log(2A_1) - 191/36]}{8A_1^4}.$$

Setting $A_1 = 10$ turns this into

$$H^{(0)}_{\text{low}} = -0.009\,97\mu^3 - 0.149\,35\mu^2 + (11/6)\mu\log(\mu) - 1.742\,17\mu. \tag{6.13}$$

6.4.3. The sum $H^{(0)}_{\text{mid}}$. Between L_1 and L_2 there are no useful approximations to h_l. Amalgamating the extra term $-\mathsf{h}_{L_1}$ with the Euler–Maclaurin formula[27], one obtains

$$H^{(0)}_{\text{mid}} = \int_{L_1}^{L_2} \mathrm{d}l\,\mathsf{h}_l + \left\{\frac{1}{2}\mathsf{h}_l + \frac{1}{12}\frac{\mathrm{d}\mathsf{h}_l}{\mathrm{d}l} - \frac{1}{720}\frac{\mathrm{d}^3\mathsf{h}_l}{\mathrm{d}l^3} + \cdots\right\}\Bigg|_{l=L_1}^{L_2}.$$

Changing the independent variable from l to α (with fixed μ) and evaluating the integral numerically turns this into

$$H^{(0)}_{\text{mid}} = P + Q(A_2) - Q(A_1) + \mathcal{O}(\mu^{-1}), \tag{6.14}$$

$$P \equiv \mu^3\frac{1}{2}\int_{A_2}^{A_1}\frac{\mathrm{d}\alpha}{\alpha^2}\left[\frac{\left(F_l^{(0)}/\nu\right)}{\alpha^2}\right] \simeq 187.176\,11\mu^3, \tag{6.15}$$

$$Q(\alpha) \equiv \mu^2\frac{1}{2}\left[\frac{\left(F_l^{(0)}/\nu\right)}{\alpha^2}\right] - \mu\frac{2}{12}\left[\alpha^2\frac{\mathrm{d}}{\mathrm{d}\alpha}\right]\left[\frac{\left(F_l^{(0)}/\nu\right)}{\alpha^2}\right], \tag{6.16}$$

$$Q(A_1) = -0.149\,35\mu^2 - 0.163\,14\mu, \qquad Q(A_2) = 48.066\,73\mu^2 + 2.462\,29\mu. \tag{6.17}$$

6.4.4. The sum $H^{(0)}_{\text{high}}$. Here α is small, and by suitably modifying the molecular calculation in section 6.2, appendix C.1 derives an auxiliary template function W which eventually yields

$$H^{(0)}_{\text{high}} = W^{(0)}(L) - W^{(0)}(L_2), \tag{6.18}$$

$$W^{(0)}(L_2) = 187.003\,24\mu^3 + 48.460\,36\mu^2 + 2.462\,23\mu, \tag{6.19}$$

$$W^{(0)}(L) = E_3X^3 + E_2X^2 + E_1X + \mathcal{O}(X^{-1}), \tag{6.20}$$

$$E_3 = \frac{64\pi^{11/4}}{5}x^{1/2} - \frac{16\pi^{13/4}}{3}x^{3/2} - 8\pi^{7/2}x^2 + 2\pi^{15/4}x^{5/2} + \frac{128\pi^3}{15}\log\left(\frac{1}{2\pi^{1/2}x}\right)x^3 - \frac{\pi^{17/4}}{2}x^{7/2},$$

$$E_2 = 2\pi^3x^2,$$

$$E_1 = \frac{7\pi^{7/4}}{2}x^{1/2} - \frac{23\pi^{9/4}}{24}x^{3/2} - \pi^{5/2}x^2 + \frac{25\pi^{11/4}}{192}x^{5/2} + \frac{52\pi^2}{45}x^3 + \frac{9\pi^{13/4}}{256}x^{7/2}.$$

Since $W^{(0)}(L)$ stems from $H^{(0)}$ in the molecular scenario, it is instructive to compare (6.20) with (6.6).

- Fractional powers of X have disappeared, and with them the peculiar powers higher than the third.

[27] The values given for $Q(A_{1,2})$ come from the exact expression for $F_l^{(0)}/\nu$. Alternatively one could first expand for high and for low α respectively, differentiate, and only then substitute for $A_{1,2}$: any differences are negligible.

- Moreover, E_3 now features $\log(x)$ (like D_3 in SF), rather than $\log(X)$ (like C_3). Since x is an intensive parameter, $W^{(0)}(L)$ is properly extensive, as it should be.
- The coefficient of X^2 has been halved: $E_2 = C_2/2$. The coefficients of X are equal: $E_1 = C_1$.
- The nonretarded limit reduces to just the two terms with $x^{1/2}$. They may be compared, provisionally, with (4.24), but will be discussed further in section 8.2.

6.4.5. Assembling $H^{(0)}$. First we substitute into (6.11) from (6.14) and (6.18) and rearrange to obtain

$$\begin{aligned} H^{(0)} &= H^{(0)}_{\rm low} + H^{(0)}_{\rm mid} + H^{(0)}_{\rm high} = H^{(0)}_{\rm low} + [P + Q(A_2) - Q(A_1)] + [-W^{(0)}(L_2) + W^{(0)}(L)] \\ &= \left\{H^{(0)}_{\rm low} - Q(A_1)\right\} + \{P + Q(A_2) - W^{(0)}(L_2)\} + W^{(0)}(L), \end{aligned} \tag{6.21}$$

where the last grouping proves the more perspicuous. On further substitution from (6.13), (6.15), (6.17) and (6.19), the end result reads

$$H^{(0)} = 0.162\,90\mu^3 - 0.393\,63\mu^2 - 1.578\,97\mu + (11/6)\mu\log(\mu) + W^{(0)}(L). \tag{6.22}$$

(In fact the component $(11/6)\mu\log(\mu)$ will be cancelled by an equal and opposite component of $H^{(1)}$.) The numbers reveal several near cancellations within each pair of braces in (6.21), striking enough to warrant further analysis in appendix C.2, if only for reassurance.

Since $W^{(0)}(L)$ unlike the other parts of $H^{(0)}$ is a function not only on μ but separately of x and X, its relative importance depends on x. If, for orientation, one sets $x \to x_C = 1.80 \times 10^{-5}$, roughly as in C_{60} and other graphiteroles, and $X = \mu/4\pi x_C$, then

$$W^{(0)}(L) = 1.092 \times 10^{11}\mu^3 + 3.927\mu^2 + 486.6\mu \qquad (\text{for } x = x_C). \tag{6.23}$$

The enormous disparity between the coefficients of μ^3 in the pseudo-molecular part $W^{(0)}(L)$ of $H^{(0)}_{\rm high}$ and in the rest of $H^{(0)}$ shows that, as regards extensive components, we can forget everything except $W^{(0)}(L)$. This should have been expected: it is a familiar fact of condensed-state physics that the binding energies of macroscopic bodies are well explained nonrelativistically in terms of instantaneous interactions between their constituents, disregarding both retardation and the quantized nature of the Maxwell field; and $W^{(0)}(L)$ is the only part of $H^{(0)}$ contributing to B in the nonretarded limit. The disparities between the coefficients of μ^2 and of μ are far smaller. It follows that such corrections to B can, indeed, be discussed only after quantizing the Maxwell field and taking full account of retardation.

Finally we note that $H^{(0)}$ is positive, as it is in the molecular scenario, and technically for much the same reason that the $F^{(0)}_l\big/\nu$ are positive for small α, whose contributions govern the dominant component $W^{(0)}(L)$.

7. The Debye expansion: orders one and two

There are two main reasons for taking the Debye expansion of H beyond order zero. First, one learns that, roughly speaking and at least initially, higher orders affect only subdominant components of H, i.e. only successively lower powers of X. For instance, order zero turns out to have given the extensive component (of order X^3) *exactly*. Second, because of the cancellation in zero order, contributions to the Boyer component start only in order 1, and to see them in any kind of perspective one needs to go at least one step beyond this.

7.1. The integrals over frequency

To find $F_l^{\mathrm{TE}(1,2)}$ one integrates $\Phi_l^{\mathrm{TE}(1,2)}$ over z, and similarly for TM. We shall quote the $F_l^{(1)}\big/\nu$ in full, to illustrate how zero and higher orders compare; but the $F_l^{(2)}\big/\nu$ are so lengthy that we shall quote only the expansions of $\left(F_l^{\mathrm{TE}(2)}\big/\nu + F_l^{\mathrm{TM}(2)}\big/\nu\right)$ for small and for large α.

To first order then, the integrands $\Phi^{(1)}/\mu^2$ are given by (A.14), (A.17). Rationalization as in zero order and very dedicated simplification (sic) eventually yield

$$\frac{1}{\nu}F_l^{\mathrm{TE}(1)} = -\frac{1}{\mu^2}\frac{1}{2\alpha^2}\left\{-\frac{1}{48}[120\pi - 240\alpha - 84\pi\alpha^2 + 128\alpha^3 - 3\pi\alpha^4 + 16\alpha^5] + \frac{[10 - 12\alpha^2 + 2\alpha^4]}{\sqrt{1-\alpha^2}}\tan^{-1}\left[\sqrt{\left(\frac{1-\alpha}{1+\alpha}\right)}\right]\right\}, \qquad (\alpha < 1), \tag{7.1}$$

with $\alpha > 1$ now requiring $\left(1/\sqrt{1-\alpha^2}\right)\tan^{-1}[\sqrt{(\cdots)}] \to \left(1/\sqrt{-1+\alpha^2}\right)\tanh^{-1}[\sqrt{-(\cdots)}]$; and

$$\frac{1}{\nu}F_l^{\mathrm{TM}(1)} = \frac{1}{\mu^2}\frac{\alpha^3}{2}\left\{\left[-2 + \frac{23\pi}{4}\alpha + 7\alpha^2 + \frac{7\pi}{2}\alpha^3\right] - \left[\tau + \upsilon\sqrt{\alpha^2+4}\right]\sqrt{\frac{\sqrt{\alpha^2+4}-\alpha}{2\alpha(\alpha^2+4)}}\tan^{-1}[\ldots] - \left[\tau - \upsilon\sqrt{\alpha^2+4}\right]\sqrt{\frac{\sqrt{\alpha^2+4}+\alpha}{2\alpha(\alpha^2+4)}}\tanh^{-1}[\ldots]\right\}, \tag{7.2}$$

where the arctan and arctanh are the same as in (6.2), and

$$\tau \equiv 7\alpha^5 + 29\alpha^3 + 18\alpha, \qquad \upsilon \equiv 7\alpha^4 + 15\alpha^2 + 2.$$

The second-order expressions feature the same arctan, arctanh and square roots, but more elaborate polynomials.

Combining the expansions of (7.1) and (7.2) for small α, one obtains

$$\frac{1}{\nu}F_l^{(1)} = \frac{1}{\mu^2}\left\{-\frac{\pi}{2}\alpha^{5/2} - \frac{17\pi}{8}\alpha^{7/2} + \frac{187\pi}{64}\alpha^4 - \frac{205\pi}{64}\alpha^{9/2} + \frac{988}{105}\alpha^5 - \frac{635\pi}{256}\alpha^{11/2} + \cdots\right\}, \tag{7.3}$$

while to second order

$$\frac{1}{\nu}F_l^{(2)} = \frac{1}{\mu^4}\left\{-\frac{\pi}{8}\alpha^{9/2} - \frac{251\pi}{32}\alpha^{11/2} + \cdots\right\}. \tag{7.4}$$

The expansions for large α read

$$\frac{1}{\nu}F_l^{(1)} = \frac{1}{\mu^2}\left\{-\frac{11}{3}\alpha^3 - \frac{3\pi}{16}\alpha^2 + \frac{3}{10}\alpha - \frac{3\pi}{64} + \mathcal{O}\left(\frac{\log(\alpha)}{\alpha}\right)\right\}, \tag{7.5}$$

$$\frac{1}{\nu}F_l^{(2)} = \frac{1}{\mu^4}\left\{-\frac{127}{30}\alpha^5 - \frac{9\pi}{1024}\alpha^4 - \frac{171\pi}{2048}\alpha^2 - \left[\frac{169\,921}{120\,120} - \frac{3}{4}\log(2\alpha)\right]\alpha + \mathcal{O}(1)\right\}. \tag{7.6}$$

The absence of α^3 from (7.6) stems from a cancellation between TE and TM, on the same footing as the cancellation of Boyer terms in zero order. Note that $F_l^{(1,2)}$ are negative at both small and large α.

7.2. Molecular scenario

Constructing $H^{(1)}$ from (7.3), one finds

$$H^{(1)} = \left\{ -\frac{\pi}{2}\mu^{1/2}S_{-1/2} - \frac{17\pi}{8}\mu^{3/2}S_{-3/2} + \frac{187\pi}{64}\mu^2 S_{-2} \right.$$
$$\left. - \frac{205\pi}{64}\mu^{5/2}S_{-5/2} + \frac{988}{105}\mu^3 S_{-3} - \frac{635\pi}{256}\mu^{7/2}S_{-7/2} + \cdots \right\}. \tag{7.7}$$

Here, since we drop terms that vanish as $X \sim L \to \infty$, the S_p are required to order L^{-p}. When $H^{(1)}$ is expanded in powers of $1/X$, it turns out that all but the term proportional to X are jokers, i.e. functions only of μ, whose coefficients will not be needed. Hence we settle for writing

$$H^{(1)} = c_1^{(1)}\mu + \boxed{\left\{ \sum_p c_p^{(1)}\mu^p \right\}} + \mathcal{O}(X^{-1}) \qquad p = 1/2, 3/2, 2, 5/2, 3, 7/2, \ldots, \tag{7.8}$$

where the $c_{p\neq 1}^{(1)}$ are pure numbers of order unity, jokers are boxed, and

$$c_1^{(1)}\mu = \left\{ -2\pi^{7/4}x^{1/2} + \frac{17\pi^{9/4}}{2}x^{3/2} - \frac{187\pi^{5/2}}{32}x^2 \right.$$
$$\left. + \frac{205\pi^{11/4}}{48}x^{5/2} - \frac{988\pi^2}{105}x^3 + \frac{127\pi^{13/4}}{64}x^{7/2} + \cdots \right\} X. \tag{7.9}$$

From (7.4) one finds similarly that

$$H^{(2)} = -\frac{\pi}{8}\mu^{1/2}S_{-5/2} - \frac{251\pi}{32}\mu^{3/2}S_{-7/2} + \cdots. \tag{7.10}$$

This is novel in that *all* these $S_{-q}(L)$ are convergent, and may be replaced by $S_{-q}(\infty)$, because, as shown in appendix B, the differences are of order L^{-q+1}, too small to affect (7.10). Hence $H^{(2)}$ consists entirely of jokers:

$$\boxed{H^{(2)} = \left\{ \sum_p c_p^{(2)}\mu^p \right\}} \qquad p = 1/2, 3/2, \cdots. \tag{7.11}$$

7.3. SF scenario

The high-α expansions (7.5), (7.6) entail

$$H^{(1)} = -\frac{11}{3}\mu S_{-1} - \frac{3\pi}{16}S_0 + \frac{3}{10}\mu^{-1}S_1 - \frac{3\pi}{64}\mu^{-2}S_2 + \mathcal{O}[\mu^{-3}(\log(\mu)S_3 + T_3)], \tag{7.12}$$

$$H^{(2)} = -\frac{127}{30}\mu S_{-3} - \frac{9\pi}{1024}S_{-2} + \mathcal{O}[\mu^{-2}S_0]. \tag{7.13}$$

These expressions may be compared with $H^{(0)}$, equation (6.7). Re-written routinely[28] in terms of x and X, they read

$$H^{(1)} = \left[-\frac{22\pi}{3}\left(-\frac{1}{2} + \gamma + \log(8\pi^{1/2}X \right) - \frac{3\pi^{3/2}}{8} + \mathcal{O}\left(\frac{1}{x}\right) \right] xX + \frac{3\pi}{16} + \mathcal{O}\left(\frac{1}{X}\right), \tag{7.14}$$

$$H^{(2)} = -\frac{254\pi}{15}\left(\frac{7}{8}\zeta(3) - 1\right)xX - \frac{9\pi}{1024}\left(\frac{\pi^2}{8} - 1\right) + \mathcal{O}\left(\frac{1}{x^2X}\right). \tag{7.15}$$

Qualitatively new are the Boyer terms, i.e. those independent of X. As claimed in section 4.3, they are nonparametric, i.e. they feature neither $\log(X)$ nor x. Section 8.3 comments on them further.

[28] That is set $\mu = 4\pi xX$, and expand the S_p for large X as in appendix B.

7.4. Macroscopic scenario

We start with $H^{(1)}$, split into three parts exactly as $H^{(0)}$ was in section 6.4.

First, $H^{(1)}_{\rm low}$ is given by the SF expression (7.12) with $L \to L_1 \equiv \mu/2A_1 - 1/2$. This produces

$$H^{(1)}_{\rm low} \simeq [5.530 - (11/6)\log(\mu)]\mu - 18.025. \tag{7.16}$$

Next, $H^{(1)}_{\rm mid}$ is turned into an integral evaluated numerically, plus Euler–Maclaurin endpoint corrections[29], proportional respectively to μ and μ^0:

$$H^{(1)}_{\rm mid} = -20.284\mu + 18.294. \tag{7.17}$$

For $H^{(1)}_{\rm high}$, we adapt the procedure for $H^{(0)}_{\rm high}$ from appendix C.1. Thus we use $H^{(1)}$ from the molecular scenario, equation (7.7), to identify a template $\tilde{W}^{(1)}(\mathcal{L})$; drop the jokers, $\tilde{W}^{(1)} \to \bar{W}^{(1)}$; substitute $\mathcal{L} \to \mu/2\mathcal{A} - 1/2$; note that here this produces no terms independent of $\mathcal{A}$, so that $\bar{W}^{(1)}$ is equally the analogue of $W^{(0)}$ itself, $W^{(1)} = \bar{W}^{(1)}$; and finally we evaluate $W^{(1)}(L_2)$ in terms of μ, while the more interesting $W^{(1)}(L)$ is expressed in terms of x and X:

$$W^{(1)}(L_2) = -3.228\mu - 0.321, \tag{7.18}$$

$$W^{(1)}(L) = \left\{-2\pi^{7/4}x^{1/2} + \frac{17\pi^{9/4}}{2}x^{3/2} - \frac{187\pi^{5/2}}{32}x^2 + \frac{205\pi^{11/4}}{48}x^{5/2} - \frac{988\pi^2}{105}x^3 + \frac{127\pi^{13/4}}{64}x^{7/2} + \cdots\right\} X + \mathcal{O}\left(\frac{1}{X}\right), \tag{7.19}$$

$$H^{(1)}_{\rm high} = W^{(1)}(L) - W^{(1)}(L_2). \tag{7.20}$$

Combining all these expressions we obtain[30]

$$H^{(1)} = H^{(1)}_{\rm low} + H^{(1)}_{\rm mid} + H^{(1)}_{\rm high} \tag{7.21}$$

$$= -(11/6)\log(\mu)\mu - 11.526\mu + 0.590 + W^{(1)}(L). \tag{7.22}$$

The analogous exercise for $H^{(2)}$ shows that $H^{(2)}_{\rm mid}$ can be discarded altogether, because it is of order μ^{-1}, while $H^{(2)}_{\rm high}$ vanishes because its template contains nothing but jokers. The sole survivor is $H^{(2)}_{\rm low}$, given by (7.13). As in (7.10), we can replace $S_{-q}(L) \to S_{-q}(\infty)$, because the differences are only of order $L^{-q+1} \sim X^{-q+1}$, whence the corrections to $H^{(2)}$ vanish as $X \to \infty$. Accordingly

$$H^{(2)} \simeq -\frac{127}{30}\mu S_{-3}(\infty) - \frac{9\pi}{1024}S_{-2}(\infty) \simeq -\frac{127}{30}\left(\frac{7\zeta(3)}{8} - 1\right)\mu - \frac{9}{4096}\left(\frac{\pi^2}{8} - 1\right). \tag{7.23}$$

Since $\zeta(3) \simeq 1.2021$, the coefficient of μ is 0.2193.

[29] The near-cancellation between the μ-independent components of $H^{(1)}_{\rm low}$ and $H^{(1)}_{\rm mid}$ reflects the fact that the Euler–Maclaurin formula would be nearly adequate if it were applied to $H^{(1)}_{\rm low}$: regarding this and other such cancellations, see appendix C.2.

[30] The Boyer component here, namely 0.590, should be equal to the second term $3\pi/16 = 0.589$ of (7.14). The apparent discrepancy by 0.001 reflects the limited numerical accuracy of (7.16), which has approximated $H^{(1)}_{\rm low}$ by just the first four terms of (7.12). In other words, the discrepancy represents an error of about 1:20000 in $H^{(1)}_{\rm low}$.

8. Discussion

8.1. Comparisons between orders 0, 1 and 2

We compare $H^{(1)}$ and $H^{(2)}$ with $H^{(0)}$, partly in order to illuminate how the Debye expansion works. The table lists the approximations defining the three scenarios, and some leading terms without regard to signs or to numerical coefficients. It ignores logarithms unless there are special reasons not to. Recall that $\mu \equiv 4\pi x X$, that $X \gg 1$ throughout, and that we drop contributions to H that would vanish as $X \to \infty$.

	molecular $x \ll 1, \mu \ll 1$	macroscopic $x \ll 1, \mu \gg 1$	SF $x \gg 1, \mu \gg 1$
$H^{(0)}$	$\{[x^{1/2}, \ldots, x^3 \log(X), \ldots]X^3$ $+[x^{1/2}, \ldots]X$ $+[\mu^{1/2}, \mu^{3/2}, \mu^2, \ldots]\}$	$\{[x^{1/2}, \ldots, x^3 \log(x) \ldots]X^3$ $+[x^{1/2}, \ldots]X$ $+[\mu^3, \mu^2, \mu \log(\mu), \mu]\}$	$\{[x \log(x), \ldots]X^3$ $+[x \log(X), \ldots]X\}$
$H^{(1)}$	$\{[x^{1/2}, \ldots]X$ $+[\mu^{1/2}, \mu^{3/2}, \mu^2, \ldots]\}$	$\{[x^{1/2}, x^{3/2}, \ldots]X$ $+[\mu \log(\mu), \mu, \ldots] + \text{const}\}$	$\{[x \log(X), \ldots]X$ $+\text{const}\}$
$H^{(2)}$	$[\mu^{1/2}, \mu^{3/2}, \mu^2, \ldots]$	$[\mu + \text{const}]$	$[x, \ldots]X + \text{const}$

- $H^{(0)}$ alone is extensive, i.e. it alone has a leading term proportional to X^3. (In the molecular scenario, where X cannot increase arbitrarily, $H^{(0)}$ contains also a small term $(xX)^3 \log(X)$.) Conversely, the zero-order term of the Debye expansion yields the thermodynamic limit of B *exactly*. (Only for convenience have we expanded it by powers of x.)
- Components proportional to X^2 enter only the molecular and macroscopic scenarios, and only through μ^2.
- As regards the shape-dependent components (proportional to X), $H^{(1)}$ is just as important as $H^{(0)}$. In the molecular scenario $H^{(2)}$ has no such component; in the macroscopic scenario its contribution is smaller than that of $H^{(1)}$ by a factor of $x^{1/2}$; in SF it is smaller by $1/\log(X)$.
- Boyer components (proportional to X^0) are considered separately below.
- By a remarkable coincidence (not registered in the table), equations (6.22) and (7.22) show that in the macroscopic scenario the terms proportional to $\mu \log(\mu)$ from $H^{(0)}$ and $H^{(1)}$ cancel. In SF, (6.8) and (7.14) indicate a corresponding cancellation between the terms proportional to $x \log(X)X$.
- In the molecular scenario, the orders of magnitude of successive orders of the Debye expansion drop in proportion to $X^3 : X : X^{1/2}$.
- In the macroscopic scenario, they drop in proportion to $X^3 : X : x^{1/2}X$.
- In SF they drop in proportion to $X^3 : X \log(X) : X$. Thus $H^{(2)}$ in SF is, uniquely, almost competitive with $H^{(1)}$.

8.2. Nonretarded limits

By construction, our model cannot correct systematically for fluid velocities that might no longer be negligible compared to c (e.g. (2.6) has dropped the Lorentz force from the start). The only relativistic effects the model admits are those due to retardation and to the quantized

nature of the Maxwell field: these are included exactly[31]. For small $x \equiv e^2/mc^2a$ (as in the molecular and macroscopic scenarios) they can be tracked through rising powers of $x^{1/2}$. By contrast, approximation proceeds by inverse powers of x in the SF scenario, which is a kind of extreme-relativistic regime, as we discussed at the end of section 3. Either way one meets logarithms, which serve as reminders that plasmas are never amenable to true perturbation theory in the sense of convergent power series.

Moreover, when operating the Debye expansion one must bear in mind that the macroscopic limit $\mu \to \infty$ and the nonretarded limit $c \to \infty$ do not commute: if $c \to \infty$ is taken first it automatically selects the molecular scenario, whereas physical interest attaches mostly to the macroscopic scenario, reached by taking $\mu \gg 1$ first, and considering $x \ll 1$ only afterwards. The prefactor c in $B = (\hbar c/4\pi R)H$ then rescues those and only those components of F_l/v or of H that are of order $\alpha^{1/2}$ or $x^{1/2}$ or $\mu^{1/2}$; all other components drop out. Fortunately the extensive term is the same whichever limit is taken first.

The nonretarded model of section 3 takes the limit $c \to \infty$ at the outset, and has the merit that no approximations of any kind are needed afterwards: our concentration on large X is a purely optional convenience. The result is $H_{\rm NR}$, displayed in (4.24). Section 4.5 has verified that the exact theory reproduces it for small enough μ. But as a matter of diagnostics it is of some interest to explore just how closely the Debye expansion can approach the exact result. The appropriate limits are the components of H proportional to $\mu^{1/2}$, visible on inspection from the basic low-α expansions (6.5), (7.7), (7.10) in orders 0, 1, 2 respectively:

$$\begin{aligned}\lim_{c\to\infty}\{H^{(0)}+H^{(1)}+H^{(2)}\} &= \pi\mu^{1/2}\left(S_{3/2}-\frac{1}{2}S_{-1/2}-\frac{1}{8}S_{-5/2}\right)\\ &= x^{1/2}\left(X^3\pi^{11/4}\left\{\left[\frac{64}{5}\right]\right\}+X\pi^{7/4}\left\{\left[\frac{7}{2}\right]-[2]\right\}\right.\\ &\quad +2\pi^{3/2}\left\{-\left[1+\zeta\left(-\frac{3}{2}\right)(2\sqrt{2}-1)\right]\right.\\ &\quad \left.\left.+\frac{1}{2}\left[1-\zeta\left(\frac{1}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)\right]+\frac{1}{8}\left[1-\zeta\left(\frac{5}{2}\right)\left(1-\frac{\sqrt{2}}{8}\right)\right]\right\}\right).\end{aligned}$$

The rightmost expression follows on expansion by powers of $1/X$, as in appendix B.3. Within each pair of braces, the successive square brackets show the contributions from orders 0, 1 and 2. Comparison with $H_{\rm NR}$ confirms that order 0 alone reproduces the true extensive term (proportional to X^3); orders 0 and 1 are both needed to reproduce the true shape-dependent term (proportional to X); the final term (independent of X) evaluates to $4\pi^{3/2}\times 0.128$. The coefficient 0.128 is close but not equal to the Abel–Plana derived $C_{\rm NR} = 0.127$; the difference probably reflects small contributions from higher orders of the Debye expansion.

Finally it is worth stressing that the NR expressions can apply even when R far exceeds the wavelength $\lambda_L = 2\pi c/\omega_L$ of light corresponding to the maximum plasmon frequency ω_L. This is important, because *a priori* a pessimistic field-theorist might well have guessed the opposite but in fact irrelevant condition $R/\lambda_L \ll 1$, entailing $x \ll \pi^{1/2}/X^2$, and out of reach of the macroscopic scenario. On the other hand, no such suspicions would have occurred to the condensed-state physicist, who knows that the cohesive energies of macroscopic bodies can be calculated very accurately without ever thinking about retardation or relativity.

[31] In atomic physics, analogous models would exclude fine structure but include the Lamb shift. Paper B.IV pursues such comparisons somewhat further.

8.3. Boyer components

Section 4.3 defined these as components H_B of H that are proportional to X^0. All those we have found are nonparametric, i.e. they feature neither $\log(X)$ nor x; equivalently, they are jokers independent of μ. It is a plausible but unproven conjecture that the exact Boyer term too is nonparametric. The molecular scenario excludes any such, simply because all its terms contain powers of x no lower than $x^{1/2}$. In the macroscopic scenario the H_B are the same as in SF, because they can occur only in the SF-related part $H_{\rm low}$ of H. As already noted, $H_B^{(0)}$ vanishes because of a cancellation between TE and TM. Though the historical interest in Boyer terms makes this the most dramatic of such cancellations, there are others on the same footing: e.g. the one in (7.6) pointed out earlier.

Here we track down $H_B^{(1)}$ and $H_B^{(2)}$, using square brackets to keep them separate. Boyer components enter only through

$$\begin{aligned}-\frac{3\pi}{16}[S_0]-\frac{9\pi}{1024}[S_{-2}] &= -\frac{3\pi}{16}[L]-\frac{9\pi}{1024}\left[\frac{3}{4}\zeta(2)-1+\mathcal{O}\left(\frac{1}{L}\right)\right]\\ &\simeq -\frac{3\pi}{16}\left[\sqrt{4\pi X^2+1}-1\right]-\frac{9\pi}{1024}\left[\frac{3}{4}\zeta(2)-1\right]\\ &\simeq -\frac{3\pi}{16}\left[2\pi^{1/2}X-1+\mathcal{O}\left(\frac{1}{X}\right)\right]-\frac{9\pi}{1024}\left[\frac{\pi^2}{8}-1\right].\end{aligned}$$

Thus $H_B^{(1)}$ comes from the -1 in $H^{(1)}$, and $H_B^{(2)}$ is all of the second term:

$$H_B = H_B^{(1)}+H_B^{(2)} = \left[\frac{3\pi}{16}\right]-\frac{9\pi}{1024}\left[\frac{\pi^2}{8}-1\right], \tag{8.1}$$

whence

$$\begin{aligned}B_B &\equiv \frac{\hbar c}{4\pi R}H_B \equiv \frac{\hbar c}{R}\times C_B,\\ C_B &\simeq \left[\frac{3}{64}\right]-\left[\frac{9}{4096}\left(\frac{\pi^2}{8}-1\right)\right] \simeq [0.0469]-[0.0005] \simeq 0.0464.\end{aligned} \tag{8.2}$$

While it seems encouraging that the second-order term is only about 1% of the first, there will of course be further contributions from higher orders of the Debye expansion. In principle we could proceed to $H_B^{(3)}$ along exactly the same lines as before, but in practice the integrations over frequency are too tedious to tackle in cold blood. Meanwhile, if one *assumes* that the Boyer terms are nonparametric, then for $n \geqslant 3$ one can get some tentative idea about the magnitude (though not the sign) of $C_B^{(n)}$ as follows. By construction, it is proportional to μ^{-2n} ; to eliminate this the Boyer component of $H_B^{(n)}$ must feature $\alpha^{2n} = (\mu/(2l+1))^{2n}$; whence $H_B^{(3)}$ must be proportional to S_{-2n+2}; which may then be replaced by $S_{-2n+2}(\infty)$, by the same arguments as in second order. Hence one might plausibly expect that

$$\left|C_B^{(n)}\right| \stackrel{?}{\sim} \mathcal{O}\left(\frac{1}{4\pi}S_{-2n+2}(\infty)\right). \tag{8.3}$$

For large n the right-hand side drops like $1/4\pi 9^{(n-1)}$. Unfortunately, this argument about pure numbers yields no clues to the proportionality constant left unspecified in (8.3), and some pessimism is induced by observing from (8.2) that for $n = 2$ it is rather small, namely $3\pi/1024 \simeq 10^{-2}$.

For comparison we record some values for C_B arrived at very differently by older methods, under the apparent misapprehensions as to its status already discussed in sections 1 and 4.3, and touched on again in appendix D. In the first evaluation, Boyer (1968) himself indicated

$C_B = 0.046$; Davies[32] (1972) has 0.046 22; Milton *et al* (1978) have 0.046 1765; Balian and Duplantier (1978) have 0.046 18; Cognola *et al* (2001) and Kirsten (2002) have 0.046 1767.

It is an open question whether higher orders of the Debye expansion would narrow the gap between these values and ours, or whether they would reveal some real discrepancy. Meanwhile, though the apparent (if only provisional) agreeement may well have some physical significance, right now one cannot tell what this might be. There are several reasons why not, and at the risk of some repetition we list the more important. First, earlier calculations were meant to address the total energy rather than the binding energy B, and there has been no convincing analysis enabling one to compare like with like. Second, the energy they consider is supposed to reside wholly in the Maxwell field, whereas part of B is kinetic energy of the electrons and localized inside the material. (Section 3 showed this for the nonretarded model, whose close relation to the dominant terms of the macroscopic scenario makes the same conclusion plausible for the latter as well.) Third, the relations between the different approaches prove the harder to explore because all nonperturbative discussions seem to have envisaged, often tacitly, that nominal divergences whatever their status would eventually be controlled by dispersion alone, whereas we have seen that physically sensible results require not only dispersion but, crucially, an appeal also to the granularity of real materials[33], implemented here by our Debye-type cutoff.

More specifically, it is instructive to compare attitudes to any Boyer components that might turn out to be parametric, say on account of coefficients increasing indefinitely with some coupling strength like x. Older approaches[34] try to reason directly in limits akin to the perfect-reflector limit $x \to \infty$; hence they would meet a divergence, and would have to treat it as a serious conceptual difficulty for renormalization theory. Just where such divergences are thought to occur depends on the dimensionality of space and on the boundary conditions imposed on the fields. In our view, however, such difficulties are illusory: parametric Boyer terms are the norm, and any associated divergences merely nominal. They raise no questions of principle: in fact it is nonparametric cases that are exceptional and possibly in need of special explanations.

To summarize, we suggest that any Boyer components that might turn out to be nominally divergent are, in real life, merely parametric; and parametric or not, by the standards of condensed-state physics their contribution to the total binding energy of macroscopic bodies is negligible.

8.4. The pressure

The principle of virtual work yields the total pressure as

$$P = -\frac{1}{4\pi R^2}\left(\frac{\partial B}{\partial R}\right)_N = \frac{\hbar c}{(4\pi)^2 R^4}\left(\frac{\partial(\mu H)}{\partial \mu}\right)_L = \frac{\hbar c}{(4\pi)^2 R^4}\left(\frac{\partial(xH)}{\partial x}\right)_X, \tag{8.4}$$

since in virtue of (2.4) fixed N means fixed L and fixed X. The penultimate expression is suited to H written as a polynomial in μ with coefficients that are functions only of L (e.g. (6.5) or (6.7)): then it is easiest to get P by differentiating first and approximating the $S_p(L)$ and $T_p(L)$ only afterwards. The last expression in (8.4) is suited to H written as a function of x and X (e.g. (6.22) with (6.20)).

The contribution from the Boyer component of H is just $P_B = \hbar c C_B/4\pi R^4$.

[32] The text of this paper is open to some confusion between radius and diameter, but the intended energy seems to be its equation (16) divided by the latter.

[33] In *perturbative* treatments of dispersive *dielectrics* this necessity is already known (B.I,II and M.I,II.), with granularity introduced as a minimum separation λ in configuration space rather than as a cutoff on wave-numbers.

[34] See footnote 3 in section 1.

Remarkably, in the nonretarded model only one-third of P is due to the action of the radial electric field on the surface charge density. The rest stems from fluid pressures acting *tangentially* within the shell, constituting a kind of mechanical surface tension. Paper B.IV analyses $P_{\rm NR}$ in detail; it also comments on the force $\mathcal{F}$ between two hemispheres produced if one cuts our original shell and separates the two halves by a distance much less than R; and on the surprising lack of any significant connection between $\mathcal{F}$ and P.

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Appendix A. The Debye expansions of the integrands

The Debye (1909b) or so-called uniform expansion is designed for Bessel functions of large order ν and large argument y, with $z \equiv y/\nu$ of order unity. It is an asymptotic expansion in powers of $1/\nu$ (though it can do remarkably well even in low orders, especially for the combinations of modified Bessel functions I and K that are our concern here). It yields expansions by powers of $1/\nu^2$ of the integrands Φ_l; and in virtue of our definition $\alpha \equiv \mu/2\nu$ the end-results emerge, very roughly speaking, as expansions by powers of μ. The basic formulae are given in sections 9.7 and 9.3 of Abramowitz and Stegun (1968).

A.1. The functions g_l

Introduce the variables $t \equiv 1/\sqrt{1+z^2}$ and $\eta \equiv 1/t + \log[zt/(t+1)]$ (irrelevantly to us, since η cancels from the combinations we shall need); and define (through recurrence relations also given by Abramowitz and Stegun) the following polynomials in t:

$$u_1 = \frac{t}{8} - \frac{5t^3}{24}, \qquad u_2 = \frac{9t^2}{128} - \frac{77t^4}{192} + \frac{385t^6}{1152},$$

$$u_3 = \frac{75t^3}{1024} - \frac{4563t^5}{5120} + \frac{17\,017t^7}{9216} - \frac{85\,085t^9}{82\,944},$$

$$u_4 = \frac{3675t^4}{32\,768} - \frac{96\,833t^6}{40\,960} + \frac{144\,001t^8}{16\,384} - \frac{7436\,429t^{10}}{663\,552} + \frac{37\,182\,145t^{12}}{7962\,624};$$

$$v_1 = -\frac{3t}{8} + \frac{7t^3}{24}, \qquad v_2 = -\frac{15t^2}{128} + \frac{33t^4}{64} - \frac{455t^6}{1152},$$

$$v_3 = -\frac{105t^3}{1024} + \frac{5577t^5}{5120} - \frac{6545t^7}{3072} + \frac{95\,095t^9}{82\,944},$$

$$v_4 = -\frac{4725t^4}{32\,768} + \frac{114\,439t^6}{40\,960} - \frac{2448\,017t^8}{245\,760} + \frac{2739\,737t^{10}}{221\,184} - \frac{40\,415\,375t^{12}}{7962\,624}.$$

Then their approximations (9.7.7–10) for $I_\nu(q)$, $K_\nu(q)$, $I'_\nu(q)$, $K'_\nu(q)$ read

$$I_\nu(q) \simeq \sqrt{\frac{t}{2\pi\nu}}\exp(\nu\eta)\left[1+\sum_{k=1}^{\infty}\frac{u_k}{\nu^k}\right], \qquad K_\nu(q) \simeq \sqrt{\frac{\pi t}{2\nu}}\exp(-\nu\eta)\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{u_k}{\nu^k}\right]; \tag{A.1}$$

$$I'_\nu(q) \simeq \frac{1}{z}\sqrt{\frac{1}{2\pi\nu t}}\exp(\nu\eta)\left[1+\sum_{k=1}^{\infty}\frac{v_k}{\nu^k}\right], \tag{A.2}$$

$$K'_\nu(q) \simeq -\frac{1}{z}\sqrt{\frac{\pi t}{2\nu}}\exp(-\nu\eta)\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{u}{\nu^k}\right]. \tag{A.3}$$

These entail

$$g_l^{\mathrm{TE}} \equiv I_\nu(q)K_\nu(q) = G_l^{\mathrm{TE}}\left\{1+\frac{c_1^{\mathrm{TE}}}{\nu^2}+\frac{c_2^{\mathrm{TE}}}{\nu^4}+\cdots\right\}, \qquad G_l^{\mathrm{TE}} = \frac{t}{2\nu}, \tag{A.4}$$

$$c_1^{\mathrm{TE}} = 2u_2 - u_1^2 = t^2/8 - 3t^4/4 + 5t^6/8, \tag{A.5}$$

$$\begin{aligned} c_2^{\mathrm{TE}} &= 2u_4 - 2u_1u_3 + u_2^2 \\ &= (t^4/128)(t^2-1)(1155t^6 - 1617t^4 + 553t^2 - 27); \end{aligned} \tag{A.6}$$

$$g_l^{\mathrm{TM}}(q) \equiv -\left\{I'_\nu K'_\nu + \frac{(I'_\nu K_\nu + I_\nu K'_\nu)}{2\nu z} + \frac{I_\nu K_\nu}{4\nu^2 z^2}\right\} \tag{A.7}$$

$$= G_l^{\mathrm{TM}}\left\{1+\frac{c_1^{\mathrm{TM}}}{\nu^2}+\frac{c_2^{\mathrm{TM}}}{\nu^4}+\cdots\right\}, \qquad G_l^{\mathrm{TM}} = \frac{1}{2\nu t z^2}, \tag{A.8}$$

$$c_1^{\mathrm{TM}} = 2v_2 - v_1^2 + t(u_1 - v_1) - t^2/4 = -(t^2/8)(1 - 6t^2 + 7t^4), \tag{A.9}$$

$$\begin{aligned} c_2^{\mathrm{TM}} &= 2v_4 - 2v_1v_3 + v_2^2 + t(u_3 - v_1u_2 + v_2u_1 - v_3) + t^2\left(u_1^2/4 - u_2/2\right) \\ &= (t^4/128)(t^2-1)(25 - 547t^2 + 1743t^4 - 1365t^6). \end{aligned} \tag{A.10}$$

The g_l progress by powers of $1/\nu^2$, because every contribution is a product of two factors with equal but opposite coefficients of odd powers of $1/\nu$.

The TM integrands require the corresponding approximation to Λ_l, namely

$$\begin{aligned} \Lambda_l &= \lim_{q\to 0}\left(q^2 g_l^{\mathrm{TM}}\right) = \nu^2 \lim_{z\to 0}\left(z^2 g_l^{\mathrm{TM}}\right) = \frac{\nu}{2}\lim_{t\to 1}\left(1+\frac{c_1^{\mathrm{TM}}}{\nu^2}+\frac{c_2^{\mathrm{TM}}}{\nu^4}+\cdots\right) \\ &= \frac{\nu}{2}\left(1-\frac{1}{4\nu^2}-\frac{0}{\nu^4}+\cdots\right) \equiv \Lambda_l^{(0)} + \Lambda_l^{(1)} + 0 + \cdots. \end{aligned} \tag{A.11}$$

Remarkably, the second correction vanishes, and the first produces agreement with the true value from (4.31):

$$\Lambda_l^{(0)} + \Lambda_l^{(1)} = \frac{\nu}{2}\left(1-\frac{1}{4\nu^2}\right) = \frac{(2\nu-1)(2\nu+1)}{8\nu} = \frac{2l(2l+2)}{8(l+1/2)} = \frac{l(l+1)}{2l+1}. \tag{A.12}$$

A.2. The integrands Φ_l

We look for expansions of the form (5.4), first for TE and then for TM.

If Φ^{TE} in (4.28) is replaced by (A.4), the result can be rearranged to read

$$\Phi^{\mathrm{TE}} = \{\log[1+\mu G^{\mathrm{TE}}] - \mu G^{\mathrm{TE}}\} + \left\{\log\left[1+\frac{\mu G^{\mathrm{TE}}}{1+\mu G^{\mathrm{TE}}}\left(\frac{c_1^{\mathrm{TE}}}{\nu^2}+\frac{c_2^{\mathrm{TE}}}{\nu^4}\right)\right] - \mu G^{\mathrm{TE}}\left(\frac{c_1^{\mathrm{TE}}}{\nu^2}+\frac{c_2^{\mathrm{TE}}}{\nu^4}\right)\right\}. \tag{A.13}$$

The first pair of braces contain the zero-order approximation $\Phi^{\mathrm{TE}(0)}$, already quoted in section 5. Orders 1 and 2 are found by expanding the contents of the second pair of braces in powers of $1/\nu$ as far as $1/\nu^4$. But in fact it proves more convenient[35] to display the results in the form (power of $1/\mu^2$) $\times$ (function of z and α):

$$\frac{1}{\mu^2}\Phi_l^{\mathrm{TE}(1)} = -\frac{\mu^2(G^{\mathrm{TE}})^2 c_1^{\mathrm{TE}}}{\nu^2(1+\mu G^{\mathrm{TE}})} = -\frac{1}{\mu^2}\frac{\alpha^4 t^4}{2(1+\alpha t)}[1-6t^2+5t^4], \tag{A.14}$$

$$\begin{aligned}\frac{1}{\mu^4}\Phi_l^{\mathrm{TE}(2)} &= -\frac{\mu^2(G^{\mathrm{TE}})^2\left(2c_2^{\mathrm{TE}}+\left(c_1^{\mathrm{TE}}\right)^2+2\mu G^{\mathrm{TE}}c_2^{\mathrm{TE}}\right)}{2\nu^4(1+\mu G^{\mathrm{TE}})^2} \\ &= -\frac{1}{\mu^4}\frac{\alpha^6 t^6}{8[1+\alpha t]^2}[(28-592t^2+2216t^4-2832t^6+1180t^8) \\ &\quad + \alpha(27t-580t^3+2170t^5-2772t^7+1155t^9)].\end{aligned} \tag{A.15}$$

For TM one proceeds as for TE, though with some aggravation on account of the terms stemming from Λ/z^2. Eventually one finds

$$\Phi_l^{\mathrm{TM}(0)} = \log[1+\mu G^{\mathrm{TM}}] - \mu G^{\mathrm{TM}} + \mu/2\nu z^2, \tag{A.16}$$

$$\begin{aligned}\frac{1}{\mu^2}\Phi_l^{\mathrm{TM}(1)} &= \frac{1}{\nu^2}\left\{\frac{\mu G^{\mathrm{TM}}c_1^{\mathrm{TM}}}{(1+\mu G^{\mathrm{TM}})} - \mu G^{\mathrm{TM}}c_1^{\mathrm{TM}}\right\} - \frac{\mu}{8\nu^3 z^2} \\ &= \frac{1}{\mu^2}\frac{\alpha^3}{2z^2[tz^2+\alpha]}[\alpha(-2+t-6t^3+7t^5)-2tz^2],\end{aligned} \tag{A.17}$$

$$\begin{aligned}\Phi_l^{\mathrm{TM}(2)} &= \frac{1}{\nu^4}\left\{\frac{\mu G^{\mathrm{TM}}c_2^{\mathrm{TM}}}{(1+\mu G^{\mathrm{TM}})} - \frac{1}{2}\left[\frac{\mu G^{\mathrm{TM}}c_1^{\mathrm{TM}}}{(1+\mu G^{\mathrm{TM}})}\right]^2 - \mu G^{\mathrm{TM}}c_2^{\mathrm{TM}}\right\} \\ &= \frac{1}{\mu^4}\frac{\alpha^6 t^3}{8z^2[tz^2+\alpha]^2}\cdot[\alpha(25-572t^2+2290t^4-3108t^6+1365t^8) \\ &\quad + z^2(24t-560t^3+2240t^5-3024t^7+1316t^9)].\end{aligned} \tag{A.18}$$

In (A.16) and in the middle expression of (A.17), the rightmost terms come from $\Lambda_l^{(0,1)}$; equation (A.18) has no such term because $\Lambda_l^{(2)}$ vanishes, as we saw at the end of section A.1.

Appendix B. The sums over multipoles

B.1. General formulae

We are concerned with the sums $S_p(L)$ and $T_p(L)$ defined in section 2, for $L \sim X \gg 1$, and recall

$$L = \sqrt{4\pi X^2+1} - 1. \tag{B.1}$$

[35] Recall the preamble to this appendix. The underlying Debye expansions re-emerge on replacing $1/\mu \to 1/2\alpha\nu$.

The sums are best expressed as

$$\begin{aligned} S_p(L) &= -1 + R_p(2L+1) - 2^p R_p(L), \\ T_p(L) &= U_p(2L+1) - 2^p U_p(L) - 2^p \log(p) R_p(L), \end{aligned} \tag{B.2}$$

where

$$R_p(n) \equiv \sum_{m=1}^{n} m^p \qquad U_p(n) \equiv \sum_{m=1}^{n} m^p \log(m). \tag{B.3}$$

The initial -1 in $S_p(L)$ stems from the absence of $l = 0$.

The asymptotic series for $R_p(n \gg 1)$ can be found either from first principles by using the Euler–Maclaurin formula, or in Hardy (1949). In our notation his result reads

$$R_p(n) = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \zeta(-p) + \sum_{r=1}(-1)^r [(-p)^{(2r-2)}] \frac{\tilde{B}_r}{(2r)!} n^{p+1-2r}, \qquad (p \neq -1), \tag{B.4}$$

where

$$[(-p)^{(M)}] \equiv (-p)(-p+1)\cdots(-p+M), \tag{B.5}$$

and the $\tilde{B}_r$ are the Bernoulli numbers[36], $(1/6, 1/30, 1/42, 1/30, 5/66)$ for $r = (1, 2, \ldots, 5)$. For positive integer p the series reduces to a polynomial. In the divergent cases $p > -1$ the dominant term of (B.4) is the first; for $p < -1$ it is the zeta function. To accommodate zeta functions with negative argument, we re-write the functional relation as

$$\zeta(-p) = -\frac{\sin(p\pi/2)}{2^p \pi^{p+1}} \Gamma(p+1)\zeta(p+1).$$

The intermediate case reads

$$R_{-1}(n) = \sum_{m=1}^{n} \frac{1}{m} = \psi(n+1) + \gamma = \log(n) + \gamma - \frac{1}{12n^2} + \frac{1}{120n^4} + \cdots. \tag{B.6}$$

The logarithmic sum is obtained by differentiation:

$$U_p(n) = \frac{\partial}{\partial p} R_p(n). \tag{B.7}$$

For convenience, the cases we meet are listed below, accurately enough for the needs of the text. Many are obtainable directly from MAPLE, without explicit use of the general formulae.

B.2. The sums as functions of L

In the macroscopic scenario, a very special role is played by those components of $S_p(L)$ and $T_p(L)$ that are independent of L: we call them *jokers*, and box them.

The divergent S_p are

$$S_5 = 16L^6/3 + 32L^5 + 220L^4/3 + 80L^3 + 127L^2/3 + 10L,$$

$$S_4 = 16L^5/5 + 16L^4 + 88L^3/3 + 24L^2 + 127L/15,$$

$$S_3 = 2L^4 + 8L^3 + 11L^2 + 6L,$$

[36] Hardy uses $t/(\mathrm{e}^t - 1) = 1 - t/2 + \sum_{r=1}^{\infty}(-1)^{r+1}\tilde{B}_r t^{2r}/(2r)!$. A popular alternative definition (MAPLE, and Abramowitz and Stegun (1968)) reads $t/(\mathrm{e}^t - 1) = \sum_{r=0}^{\infty} B_r t^r/r!$, so that $\tilde{B}_r = (-1)^{r+1} B_{2r}$.

$$S_2 = 4L^3/3 + 4L^2 + 11L/3, \qquad S_1 = L^2 + 2L, \qquad S_0 = L,$$
$$S_{-1} = \frac{1}{2}\psi\left(L + \frac{3}{2}\right) - 1 + \frac{\gamma}{2} + \log(2) \simeq \frac{1}{2}\log(L) + \boxed{\{-1 + \log(2) + \gamma/2\}} + \frac{1}{2L} - \frac{11}{48L^2} + \frac{1}{8L^3},$$

$$S_{3/2} \simeq \frac{4\sqrt{2}}{5}L^{5/2} + 2\sqrt{2}L^{3/2} + \frac{11\sqrt{2}}{8}L^{1/2} + \boxed{\{-1 - \zeta(-3/2)(2^{3/2} - 1)\}} + \frac{3\sqrt{2}}{16}L^{-1/2},$$

$$S_{1/2} \simeq \frac{2\sqrt{2}}{3}L^{3/2} + \sqrt{2}L^{1/2} + \boxed{\{-1 - \zeta(-1/2)(2^{1/2} - 1)\}} + \frac{11\sqrt{2}}{48}L^{-1/2} - \frac{\sqrt{2}}{32}L^{-3/2},$$

$$S_{-1/2} \simeq 2^{1/2}L^{1/2} + \boxed{\{-1 + \zeta(1/2)(1 - 2^{-1/2})\}} + \frac{\sqrt{2}}{2}L^{-1/2} - \frac{11\sqrt{2}}{96}L^{-3/2}.$$

The convergent S_p are

$$S_{-2} \simeq \boxed{\{-1 + 3\zeta(2)/4\}} - \frac{1}{4}L^{-1} + \frac{1}{4}L^{-2}, \qquad S_{-3} \simeq \boxed{\{-1 + 7\zeta(3)/8\}} - \frac{1}{16}L^{-2} + \frac{1}{8}L^{-3};$$

$$S_{-3/2} \simeq \boxed{\{-1 + \zeta(3/2)(1 - 2^{-3/2})\}} - \frac{\sqrt{2}}{2}L^{-1/2} + \frac{\sqrt{2}}{4}L^{-3/2},$$

$$S_{-5/2} \simeq \boxed{\{-1 + \zeta(5/2)(1 - 2^{-5/2})\}} - \frac{\sqrt{2}}{12}L^{-3/2} + \frac{\sqrt{2}}{8}L^{-5/2},$$

$$S_{-7/2} \simeq \boxed{\{-1 + \zeta(7/2)(1 - 2^{-7/2})\}} - \frac{\sqrt{2}}{40}L^{-5/2} + \frac{\sqrt{2}}{16}L^{-7/2}.$$

Regarding S_{-2} we recall $\{-1 + 3\zeta(2)/4\} = \{\pi^2/8 - 1\}$.

The logarithmic sums occur in the SF scenario. The text needs only $T(1)$ to order L^{-1} and $T(5)$ to order L^3:

$$T_5 \simeq \left[\frac{16}{3}\log(2L) - \frac{8}{9}\right]L^6 + [32\log(2L)]L^5 + \left[\frac{220}{3}\log(2L) + \frac{44}{3}\right]L^4 + [80\log(2L) + 36]L^3,$$

$$T_1 \simeq \left\{\left[\log(2L) - \frac{1}{2}\right]L^2 + [2\log(2L)]L + \frac{11}{12}\log(L) + \boxed{\{(13/12)\log(2) + 11/12 + \zeta'(-1)\}} + \frac{1}{4}L^{-1}\right\},$$

where $\zeta'(z) \equiv \mathrm{d}\zeta(z)/\mathrm{d}z$.

B.3. The sums as functions of X

Ultimately we must approximate these sums for $X \gg 1$, after substituting for L from (B.1). Then, with jokers no longer boxed, they read as follows.

$$S_5 = \frac{1024\pi^3}{3}X^6 + \frac{448\pi^2}{3}X^4 + 20\pi X^2, \qquad S_3 = 32\pi^2X^4 + 12\pi X^2, \qquad S_1 = 4\pi X^2;$$

$$S_4 \simeq \frac{512\pi^{5/2}}{5}X^5 + \frac{128\pi^{3/2}}{3}X^3 + \frac{74\pi^{1/2}}{15}X - 1 + \frac{7\pi^{-1/2}}{60}X^{-1} - \frac{\pi^{-3/2}}{480}X^{-3},$$

$$S_2 \simeq \frac{32\pi^{3/2}}{3}X^3 + \frac{10\pi^{1/2}}{3}X - 1 + \frac{\pi^{-1/2}}{6}X^{-1} - \frac{\pi^{-3/2}}{192}X^{-3},$$

$$S_0 \simeq 2\pi^{1/2}X - 1 + \frac{\pi^{-1/2}}{4}X^{-1} - \frac{\pi^{-3/2}}{64}X^{-3},$$

$$S_{-1} \simeq \frac{1}{2}\log(8\pi^{1/2}X)) + \frac{\gamma}{2} - 1 + \frac{13\pi^{-1}}{192}X^{-2} - \frac{287\pi^{-2}}{307\,20}X^{-4},$$

$$S_{-2} \simeq \left(\frac{3}{4}\zeta(2) - 1\right) - \frac{\pi^{-1/2}}{8}X^{-1} + \frac{7\pi^{-3/2}}{384}X^{-3},$$

$$S_{-3} \simeq \left(\frac{7}{8}\zeta(3) - 1\right) - \frac{\pi^{-1}}{64}X^{-2} + \frac{5\pi^{-2}}{1024}X^{-4};$$

$$S_{3/2} \simeq \frac{32\pi^{5/4}}{5}X^{5/2} + \frac{7\pi^{1/4}}{4}X^{1/2} - 1 - \zeta\left(-\frac{3}{2}\right)(2^{3/2} - 1) + \frac{713\pi^{-3/4}}{15\,360}X^{-3/2},$$

$$S_{1/2} \simeq \frac{8\pi^{3/4}}{3}X^{3/2} - 1 - \zeta\left(-\frac{1}{2}\right)(2^{1/2} - 1) + \frac{23\pi^{-1/4}}{48}X^{-1/2} - \frac{291\pi^{-5/4}}{20\,480}X^{-5/2},$$

$$S_{-1/2} \simeq 2\pi^{1/4}X^{1/2} - 1 + \zeta\left(\frac{1}{2}\right)(1 - 2^{-1/2}) + \frac{25\pi^{-3/4}}{192}X^{-3/2} - \frac{631\pi^{-7/4}}{49\,152}X^{-7/2},$$

$$S_{-3/2} \simeq -1 + \zeta\left(\frac{3}{2}\right)(1 - 2^{-3/2}) - \frac{\pi^{-1/4}}{2}X^{-1/2} + \frac{9^{-5/4}}{256}X^{-5/4},$$

$$S_{-5/2} \simeq -1 + \zeta\left(\frac{5}{2}\right)(1 - 2^{-5/2}) - \frac{\pi^{-3/4}}{24}X^{-3/2} + \frac{29\pi^{-7/4}}{3072}X^{-7/2},$$

$$S_{-7/2} \simeq -1 + \zeta\left(\frac{7}{2}\right)(1 - 2^{-7/2}) - \frac{\pi^{-5/4}}{160}X^{-5/2} + \frac{31\pi^{-9/4}}{12\,288}X^{-9/2}.$$

In the S_p it is worth tracking the terms independent of X. For odd positive integer p there are none. For nonnegative even integer p they are -1; for negative integer $p \leqslant -2$ and for all half-integer p they combine -1 with ζ functions.

Finally

$$T_5 \simeq \left\{\pi^3X^6\left[-\frac{512}{9} + \frac{1024}{3}\log(4\pi^{1/2}X)\right] + \pi^2X^4\left[-\frac{64}{3} + \frac{448}{3}\log(4\pi^{1/2}X)\right] + \pi X^2\left[-\frac{31}{45} + 20\log(4\pi^{1/2}X)\right]\right\},$$

$$T_1 \simeq \pi X^2[-2 + 4\log(4\pi^{1/2}X)] + \left[-\frac{1}{12} + 2\log(2) + \frac{11}{12}\log(\pi X) + \zeta'(-1)\right] + \frac{593\pi^{-1}}{11\,520}X^{-2}.$$

Appendix C. Macroscopic scenario in zero order

We spell out details of the derivation of $H^{(0)}_{\text{high}}$ that were skipped in section 6.4.4, and then analyse the remarkable internal cancellations encountered in the construction of $H^{(0)}$, which section 6.4.5 merely mentioned in passing.

C.1. Derivation of $H^{(0)}_{\rm high}$

Here α is small, and we exploit the results from the molecular scenario, modifying them however in two critical steps. It proves convenient to introduce, for use as placeholders at fixed μ, auxiliary script variables $(\mathcal{L}, \mathcal{A})$, interrelated like (α, l) and (A, L):

$$\mathcal{L} = \mu/2\mathcal{A} - 1/2. \tag{C.1}$$

We are concerned now with large μ and small $\mathcal{A}$.

Start from the obvious fact that

$$H^{(0)}_{\rm high} = \tilde{W}^{(0)}(L) - \tilde{W}^{(0)}(L_2), \qquad \tilde{W}^{(0)}(\mathcal{L}) \equiv \sum_{l=1}^{\mathcal{L}} \mathsf{h}_l. \tag{C.2}$$

The *template* $\tilde{W}^{(0)}(\mathcal{L})$ is got from the basic low-α expansion (6.5) by replacing $S_p(L) \to S_p(\mathcal{L})$.

The first critical step is to change the template, $\tilde{W}^{(0)} \to \bar{W}^{(0)}$, by dropping from the S_p all jokers, i.e. all functions of μ alone: this leaves $H^{(0)}_{\rm high}$ unchanged, i.e. $H^{(0)}_{\rm high} = \bar{W}^{(0)}(L) - \bar{W}^{(0)}(L_2)$, because the jokers would cancel from the difference (C.2) anyway.

Next, $\bar{W}^{(0)}$ is expressed in terms of $\mathcal{A}$ by using (C.1), and then re-expanded for large μ (appropriate macroscopically, though molecularly it would not be). Remarkably, this produces some further components that are independent of $\mathcal{A}$ and thereby functions only of μ, namely

$$\Delta\bar{W}^{(0)} = (2/15)\log(\mu)\mu^3 + (\pi/8)\mu^2.$$

The second critical step is to drop $\Delta\bar{W}^{(0)}$, i.e. to replace $\bar{W}^{(0)} \to W^{(0)} \equiv \bar{W}^{(0)} - \Delta\bar{W}^{(0)}$, which leaves $H^{(0)}_{\rm high}$ unchanged as did the first step. (In other words, though $\Delta\bar{W}^{(0)}$ is not a joker by birth, at this point it functions just as if it were.) The result reads

$$\begin{aligned} W^{(0)}(\mathcal{A}) = \Bigg\{ &\left[\frac{\pi}{5}\mathcal{A}^{-5/2} - \frac{\pi}{12}\mathcal{A}^{-3/2} - \frac{\pi}{8}\mathcal{A}^{-1} + \frac{\pi}{32}\mathcal{A}^{-1/2} + \frac{2}{15}\log(\mathcal{A}) - \frac{\pi}{128}\mathcal{A}^{1/2} + \cdots\right]\mu^3 \\ &+ \left[\frac{\pi}{2}\mathcal{A}^{-3/2} - \frac{\pi}{8}\mathcal{A}^{-1/2} + \frac{\pi}{64}\mathcal{A}^{1/2} + \frac{2}{15}\mathcal{A} + \frac{\pi}{256}\mathcal{A}^{3/2} + \cdots\right]\mu^2 \\ &+ \left[\frac{\pi}{4}\mathcal{A}^{-1/2} - \frac{\pi}{48}\mathcal{A}^{1/2} - \frac{\pi}{384}\mathcal{A}^{3/2} - \frac{2}{45}\mathcal{A}^2 - \frac{\pi}{512}\mathcal{A}^{5/2} + \cdots\right]\mu \Bigg\} + \mathcal{O}(\mu^{-1}), \end{aligned} \tag{C.3}$$

$$H^{(0)}_{\rm high} = W^{(0)}(\mathcal{A} = A(L)) - W^{(0)}(\mathcal{A} = A(L_2)) \equiv W^{(0)}(L) - W^{(0)}(L_2), \tag{C.4}$$

as in (6.18).

Finally one must evaluate $W^{(0)}(L)$ and $W^{(0)}(L_2)$. Setting $\mathcal{A} \to A_2 = 0.1$ yields (6.19) for $W^{(0)}(L_2)$. The other part $W^{(0)}(L)$ is found by setting $\mathcal{A} \to A$ from (6.9), and re-expanding by inverse powers of X. The procedure is the same as in the molecular scenario, but now it is applied to $W^{(0)}$ instead of $H^{(0)}$. It yields (6.20).

C.2. The cancellations in $H^{(0)}$

We consider the two pairs of braces in (6.21), and note as a preliminary that the numbers from (6.13) and (6.17) show that the contents of the first pair are relatively small, except perhaps for the term with $\mu\log(\mu)$.

Now consider the second pair of braces, taking P, $Q(A_2)$ and $W^{(0)}(L_2)$ from (6.15), (6.17) and (6.19):

$$\begin{aligned} [P] + [Q(A_2)] - [W^{(0)}(L_2)] = {} & [187.176\,11\mu^3] + [48.066\,73\mu^2 + 2.462\,29\mu] \\ & - [187.003\,24\mu^3 + 48.460\,36\mu^2 + 2.462\,23\mu]. \end{aligned}$$

The obvious near-cancellation between the large terms of order μ^3 implies (since the first pair of braces contains very little μ^3) that $-W^{(0)}(L_2)$ almost cancels $H^{(0)}_{\rm low} + H^{(0)}_{\rm mid}$. The reason is that, by construction, $W^{(0)}(L_2)$ is just a flawed estimate of the contributions from the multipoles with $l < L_2$. It is flawed because it represents *all* such multipoles as if they had $\alpha \ll 1$; but the flaw is only slight, because only a small fraction has $\alpha \gtrsim 1$; the true contribution from these modes is $H^{(0)}_{\rm low} + H^{(0)}_{\rm mid}$; hence, but for the flaw, the cancellation would be complete.

Like the cancellations to order μ^3, those to orders μ^2 between $H^{(0)}_{\rm low}$ and $Q(A_1)$, and those to orders μ^2 and μ between $W^{(0)}(L_2)$ and $Q(A_2)$, are neither exact nor coincidental. To see this, imagine approximating $H^{(0)}_{\rm low}$ and $H^{(0)}_{\rm high}$ through the Euler–Maclaurin formula, as $H^{(0)}_{\rm mid}$ has been. The integrals, proportional to μ^3, would depend on the behaviour of the integrand everywhere between the integration limits. But the first and the second corrections, proportional to μ^2 and to μ respectively, would depend only on the integrand and its derivative precisely at the limits; and they would automatically be equal and opposite on the two sides, i.e. the cancellations in question would be exact. Between $H^{(0)}_{\rm low}$ and $Q(A_1)$, the reason why they are not exact seems to be that the Euler–Maclaurin formula is not exact. In particular, the formula cannot produce a term proportional to $\mu \log(\mu)$ such as is certainly present in $H^{(0)}_{\rm low}$, and this inadequacy appears to have infected[37] the linear term that the formula does produce. The discrepancies between $H^{(0)}_{\rm high}$ and $Q(A_2)$, though they feature no logarithms, are somewhat larger; they appear to be caused by differences between $W^{(0)}$ and $H^{(0)}_{\rm high}$ reflecting our modifications of the template, $\tilde{W}^{(0)} \to W^{(0)}$.

It is a technical misfortune that these near-cancellations seem incapable of supplying a basis for approximations directly to $H^{(0)}$.

Appendix D. On orthodox renormalization theory for fields experiencing potentials

For the plasma model defined in section 2, we have derived sensible results by imposing the Debye cutoff, and then working with finite values of the coupling-strength parameter x: recall in particular the comments on nominal divergences in section 4.3. All this hinges on the fact that the plasma is dispersive, in virtue of dynamical degrees of freedom (from $\boldsymbol{\xi}$ and $\boldsymbol{\Pi}$) additional to those of the Maxwell field.

To obviate confusion, this approach needs to be compared briefly with a different but partly overlapping recent critique of older methods, a critique based on the orthodox theory of renormalizable quantum fields, and designed to exhibit just what divergences such a theory can or cannot remove.

By *older* we mean methods invented specially for Casimir problems, such as those referenced in section 8.3, and used recently by Milton (2003). (See also Bordag *et al* (1999), who use heat-kernel methods for fields coupled to potentials proportional to $\delta(r - R)$, i.e. for a system formally akin to our TE modes.) The critique is formulated by Graham *et al* (2001, 2002a, 2002b, 2003a), Jaffe (2003), and in a particularly accessible way by Graham *et al* (2003b). We discuss only their conclusions for spherical shells.

They consider a scalar field, experiencing a potential exerted by material with no degrees of freedom of its own. By virtue of the isomorphy noted just below (4.10), this admits analogies with the TE but not with the TM contributions in our model. They start by using standard diagrammatic methods to renormalize the total energy minus that of empty space, for material with finite overall coupling strength occupying a finitely thick shell. By construction,

[37] It is ironical that $\mu \log(\mu)$ seems to create awkwardness in $H^{(0)}$ but cancels from $H^{(0)} + H^{(1)}$.

the energy so renormalized is finite, and all its components are observable at least in principle: the crucial point is the converse, that there is no possibility of renormalizing away any further divergences that further would-be limits or approximations might produce. They consider two such limits. In the first the potential becomes proportional to $\lambda\delta(r-R)$, analogously to what our model has from the start. The second limit is $\lambda\to\infty$, which imposes Dirichlet boundary conditions on the field at $r=R$, analogous to the perfect-reflector conditions ensuing from $\mu\to\infty$ in our section 4.1. This produces divergences, ascribed to the unphysical nature of the limit.

Thus their conclusions regarding physically significant expressions for the energy are much the same as ours. On the other hand, though the two theories agree about the irrelevance of nominal divergences, they have different priorities in dealing with the real ones[38]. The plasma model is perhaps the more realistic and certainly the more ambitious, but it is not renormalizable, whence its mathematics is not anchored to any rigorous theory: that is why it has to secure convergence through a Debye cutoff, and why it fails to generalize readily to other fields and to other dimensionalities. By contrast, the mathematics of the renormalizable model is impeccable, but the model uses an interaction very different from electromagnetism: that is why it cannot yet say much about B for real materials, neither about the extensive nor about the subdominant Boyer components.

References

Abramowitz M and Stegun I (ed) 1968 *Handbook of Mathematical Functions* (New York: Dover)
Ashcroft N W and Mermin N D 1976 *Solid State Physics* (New York: Holt, Rinehart & Winston) chapter 20
Balian R and Duplantier B 1978 *Ann. Phys., NY* **112** 165
Barnett S M, Huttner B, Loudon R and Matloob R 1996 *J. Phys. B: At. Mol. Opt. Phys.* **29** 3763
Barton G 1985 *J. Phys. A: Math. Gen.* **18** 479
Barton G 1989 *Phys. Rev.* D **40** 4096
Barton G 2001a *J. Phys. A: Math. Gen.* **34** 4083 referred to as B.I
Barton G 2001b *J. Phys. A: Math. Gen.* **34** 5781
Barton G 2002 *Int. J. Mod. Phys.* A **17** 767 referred to as B.II
Barton G 2003a, b in preparation, referred to as B.IV, B.V
Barton G and Eberlein C 1991 *J. Chem. Phys.* **95** 1512
Bordag M, Kirsten K and Vassilevich D 1999 *Phys. Rev.* D **59** 085011
Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
Bouwkamp C J and Casimir H B G 1954 *Physica* **20** 539
Boyer T H 1968 *Phys. Rev.* **174** 1764
Candelas P 1982 *Ann. Phys., NY* **143** 241
Cognola G, Elizalde E and Kirsten K 2001 *J. Phys. A: Math. Gen.* **34** 7311
Davies B 1972 *J. Math. Phys.* **13** 1824
Debye P 1909a *Ann. Phys., Lpz* **30** 57
Debye P 1909b *Math. Ann.* **47** 535
Debye P 1912 *Ann. Phys., Lpz* **39** 789
Erdélyi A *et al* (ed) 1953 *Higher Transcendental Functions* vol II (New York: McGraw-Hill)
Fetter A L 1973 *Ann. Phys., NY* **81** 367
Fetter A L 1974 *Ann. Phys., NY* **88** 1
Fetter A L and Walecka J D 1971 *Quantum Theory of Many-Particle Systems* (New York: McGraw-Hill)
Gerchikov L G, Efimov P V, Mikoushkin V M and Solov'yov A V 1998 *Phys. Rev. Lett.* **81** 2707
Goldberger M L and Watson K M 1964 *Collision Theory* (New York: Wiley)

[38] The consequent technical differences are illustrated by comparing expressions for (the TE part of) B. Our model constructs this from $\sum(1/n)\mathcal{I}_n$, where $\mathcal{I}_n\equiv\sum_{l=1}^{L}\int_0^\infty \mathrm{d}y(2l+1)\,[\mu I_\nu(y)K_\nu(y)]^n$, as can be seen by expanding the logarithms in (4.28). The integral converges but the sum without the cutoff L would not. Now consider, as in section 5 of Graham *et al* (2003b), the renormalizable model with a cutoff on the Fourier transform of a radial density proportional to $\lambda\delta(r-R)$. Then $\mathcal{I}_n$ is constructed from the same summand/integrand (with λ in place of μ), but with the limits $\int_0^Y \mathrm{d}y\sum_{l=0}^{\infty}\ldots$, where the sum over l converges but the integral without the cutoff Y would not.

Gottfried K 1966 *Quantum Mechanics* section 15 (New York: Benjamin)
Graham N, Jaffe R L, Khemani V, Quandt M, Scandurra M and Weigel H 2002a *Nucl. Phys.* B **645** 49
Graham N, Jaffe R L and Weigel H 2002b *Int. J. Mod. Phys.* **17** 846
Graham N, Jaffe R L, Quandt M and Weigel H 2001 *Phys. Rev. Lett.* **87** 131601
Graham N, Jaffe R L, Khemani V, Quandt M, Scandurra M and Weigel H 2003a *Phys Lett.* B **572** 196
Graham N, Jaffe R L, Khemani V, Quandt M, Schröder O and Weigel H 2003b *Preprint* hep-th/0309130 v1
Hagen C R 2000 *Phys. Rev.* D **61** 065005
Hagen C R 2002 *Phys. Lett.* A **300** 591
Hardy G H 1949 *Divergent Series* (Oxford: Clarendon) section 13.10
Huttner B and Barnett S M 1992 *Phys. Rev.* A **46** 4306
Iglesias-Groth S, Ruiz A, Bretón J and Gomez Llorente J M 2002 *J. Chem. Phys.* **116** 10648
Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley)
Jaffe R L 2003 *Preprint* hep-th/0307014 v2
Ju N, Bulgac A and Keller J W 1993 *Phys. Rev.* B **48** 9071
Kirsten K 2002 *Spectral Functions in Mathematics and Physics* (Boca Raton, FL: Chapman and Hall/CRC)
Marachevsky V N 2001a *Phys. Script.* **64** 205 referred to as M.I
Marachevsky V N 2001b *Mod. Phys. Lett.* A **16** 1007 referred to as M.II
Milton K A 2003 *Phys. Rev.* D **68** 065020
Milton K A, DeRaad L L and Schwinger J 1978 *Ann. Phys., NY* **115** 388
Newton R G 1982 *Scattering Theory of Waves and Particles* 2nd edn (New York: Springer)
Renne M J 1971 *Physica* **53** 193 and **56** 125
Taylor R 1995 *The Chemistry of Fullerenes* (Singapore: World Scientific) p 49
Vasvári B 1996 *Z. Phys.* B **100** 223
Watson G N 1944 *A Treatise on the Theory of Bessel Functions* 2nd edn (Cambridge: Cambridge University Press)